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ELECTROMAGNETIC SCATTERING BY PERFECTLY CONDUCTING OPEN SURFACE--ETC(U)
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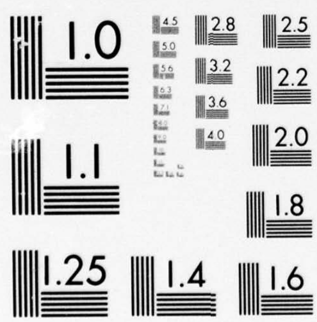

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ELECTROMAGNETIC SCATTERING BY PERFECTLY CONDUCTING OPEN SURFACES,

10 John S./Asvestas

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18 AFOSR 19 TR-77-Ø626 17 A4

11 March 1977 12 68p.

15 ✓AF
Prepared under AFOSR [redacted]-74-2634 - 74

for

Air Force Office of Scientific Research
Bolling Air Force Base, D.C.

Program Manager

R. N. Buchal

Directorate of Mathematical and Information Sciences

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Foreword

The purpose of this report is to examine the problem of scattering of electromagnetic waves by an open, perfectly conducting surface. Specifically, to formulate the problem as a boundary value problem, to convert the latter into a problem in integral equations, and to show that it can have at most one solution. In trying to reach these objectives we were guided by the methods employed to achieve the same ends for the problem of scattering by a closed surface. In this latter case there are two items of special concern: the types of surfaces and the types of linear current densities induced on these surfaces that would allow for a successful completion of the aforementioned tasks. In the case of an open surface these same items appear but in a more complicated form: in the interior of the surface things are not any different from a closed surface; near the edge, however, additional requirements must be imposed both on the surface and on the induced linear current density. Though it is not immediately apparent from reading this report, much of the time devoted to its preparation was spent in finding the right class of open surfaces and current densities for which the objectives could be accomplished.

For convenience the report has been divided into two parts. In Part II we deal with problem described above. In Part I we examine the behavior, near the edge of the open surface, of the quasi-stationary form of the integral representations for the scattered fields obtained in Part II. This part is essential in completing the proof that the boundary value problem is equivalent to a problem in integral equations. The apparent reversal in the order of the two parts is intentional, for it is in what we call Part I that all the restrictions on the surface and some on the current density come into play. The other possible arrangement, i.e. making Part I an appendix of the main problem, would necessitate bringing in these restrictions at the beginning of the report without any possible explanation. Again for convenience, we have supplied each part with its own title, abstract, and introduction.

PART I

The Behavior of Potentials with Singular Densities Near the Edge of an Open Surface.

Abstract

This work deals with the behavior of a simple-layer potential and its spatial derivatives near the edge of an open surface. Types of open surfaces and density distributions, singular near the boundary of such surfaces, are determined for which these potentials satisfy a finiteness of energy condition. The resulting estimates are useful in establishing integral representations for the electromagnetic fields scattered by a perfectly conducting open surface.

A. Introduction.

It has long been recognized (Jones, 1964; Sommerfeld, 1964) that the conditions required for scattering problems involving closed surfaces to be well-posed are not sufficient in case the surfaces are open. In fact, it has been shown (Jones, 1964) that in order that the open surface scattering problem have at most one solution (if at all), then one more condition is required of the scattered fields, be they acoustic or electromagnetic. This condition requires that the fields behave in a specified way in the vicinity of the edge of the open surface and is usually referred to as an edge condition. Though a condition of this type is dictated purely from the mathematics of the problem, it has also a physical meaning: It is intimately connected to what is known as the finiteness of energy condition which requires that, in a volume devoid of sources, the energy content should go to zero with the volume.

In this work we will examine whether a simple-layer potential and its first derivatives satisfy such an edge or energy condition. This question came about from studying the problem of finding integral representations for the electromagnetic fields scattered by a perfectly conducting open surface. Once found, these representations, which are in effect the Stratton-Chu formulas, must be tested as to whether they satisfy an edge condition. It can be shown that this can be accomplished by examining a simple-layer potential whose density, defined on the open surface, is allowed to increase beyond bound (though in a specified way) in the approach to the boundary of the surface. Specifically, we will show that the line integral of the square of the potential, and also of its spatial derivatives, around a circle centered on the boundary of the open surface and lying on a plane perpendicular to the boundary exhibit a certain order behavior with respect to the radius of the circle. In Section B we will introduce the types of open surfaces to be considered. In Section C we will describe the class of allowable densities and will state the problem and the results. In Sections D through G we will prove the results stated in Section C, while in Section H we will offer some concluding remarks. Some detailed computations will be left for Appendices A through D.

Before closing we mention that problems of this type have been studied in great detail in two dimensions for Cauchy-type integrals whose density is defined on an open curve (Muskhelishvili, 1953; Gakhov, 1966). As above, the density is allowed to grow beyond bound near the end-points of the curve. Using these results, Hayashi (1973) was able to obtain order relations near the end-points of the curve for the scattered fields for the two-dimensional Dirichlet problem for the Helmholtz equation. Some criticism of his results is offered in Section H.

B. Description of the surface.

In this section we will introduce the type of surface over which the density of the simple layer potential will be defined. In order to do so we need a number of definitions, some of which we will adopt from Carmo (1976).

Definition 1. A surface S in R^3 is a smooth surface if at each point M of S there exists a unique tangent plane (and, hence, a unique normal line) and a positive number d, the same for all points M of S, such that if we erect a rectangular coordinate system with origin at M and the z-axis along the normal at M, then the portion of S intercepted by the sphere $x^2+y^2+z^2 \leq d^2$ can be represented in the form

$$z = F(x,y), \quad (x,y) \in \Lambda'; \quad F(0,0) = \frac{\partial F(0,0)}{\partial x} = \frac{\partial F(0,0)}{\partial y} = 0, \tag{1}$$

where F is an one-to-one function with continuous second partials in Λ', and where Λ' is the closed region of the xy-plane which is the projection onto the xy-plane of the portion of S intercepted by the sphere.

With respect to this definition we note that the one-to-one property of F guarantees that S is not self-intersecting, which in turn guarantees the uniqueness of the tangent plane at each point of S. Moreover, since there are two choices of a unit normal on $z = F(x,y)$, we choose the one whose z-component points in the same direction as the positive z-axis.

Definition 2. A surface S in R^3 is a smooth open surface if

(a) S is a smooth surface

(b) for any two points in R^3 which do not belong to S there exists at least one continuous curve connecting them which does not have any points in common with S.

(c) S has as its boundary a simple closed and rectifiable curve C.

Definition 3. A smooth surface S is an orientable surface if it is possible to cover it with a family of coordinate neighborhoods as in (1) in such a way that if a point M of S belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at M.

This definition says in effect that, at each point in the common part of two overlapping neighborhoods, the normals for each neighborhood point in the same direction.

Definition 4. A surface S in R^3 is a bounded surface if there exists a sphere that contains S in its interior.

Definition 5. A surface S in R^3 is a connected surface if any two of its points can be connected by a continuous curve on S.

<u>Definition 6</u>. A surface S in R^3 is a <u>simply connected surface</u> if it is connected and if every simple closed curve on S can be continuously deformed into a point of S.

The type of surface we are interested in is a bounded, simply connected, orientable, smooth open surface with some additional conditions on its boundary as well as the portion of the surface near the boundary. The condition of boundedness together with that of smoothness result in the surface having area (cf.Carmo, 1976). The condition of connectedness is not really necessary. The actual surface may be composed of a (finite) number of disjoint surfaces. Each such surface again could be multiply connected provided that the "holes" in it are not composed of a single point in R^3, and that the curves bounding these "holes" as well as the surface near them obey the conditions described below. As it will become clear later on, however, an extension of the ensuing results to such surfaces is immediate, and for this reason, as well as to avoid cumbersome notations, we restrict ourselves to a simply connected surface.

We now turn to the description of the boundary as well as of the surface near it. From this point on we call the surface S, its boundary C, and its closure $\bar{S}$ ($\bar{S} = S \cup C$).

The properties of the previous paragraph apply now to $\bar{S}$ and not just S. Thus, if M is a point of C, we have the representation (1) for the portion of $\bar{S}$ inside the sphere of radius d with center at M. We require next that at each point of C there exists a well-defined tangent line which we identify with the x-axis of the local coordinate system. We take the x-axis to be positively oriented with respect to the z-axis, so that the y-axis is pointed towards S. With respect to this coordinate system, we impose the following condition on $\bar{S}$:

For all points M of C there exist fixed positive numbers A, a', and b' such that

$$|F(x,y)-F(0,y)| \leq A|x|^{1+\mu}, \quad |x| \leq a' < d, \quad 0 \leq y \leq b' < d \qquad (2)$$

where $\mu \geq \tilde{\mu} > 0$ and may depend on the point M of C.

This condition can be given a geometric interpretation. The unit normal vector for the portion of the surface under consideration is given by

$$\hat{n} = \frac{-F_x\hat{x}-F_y\hat{y}+\hat{z}}{\sqrt{1+F_x^{\ 2}+F_y^{\ 2}}} \quad .$$

Since from (2) $F_x(0,y) = 0$, $0 \leq y \leq b' < d$, we have that on the yz-plane, the unit normal is

$$\hat{n} = \frac{-F_y(0,y)\hat{y}+\hat{z}}{\sqrt{1+F_y^2(0,y)}}$$

But this is precisely the unit normal vector to the two-dimensional curve $z = F(0,y)$, $0 \le y \le b' < d$, provided its arclength is increasing with y. Thus, Eq. (2) requires that the three-dimensional curve $z = F(0,y)$ behaves as a two-dimensional curve in $0 \le y < b' < d$, which in turn says that the curve has torsion zero, (or, equivalently, its osculating plane is the yz-plane). This lack of torsion can be loosely restated in terms of the surface itself as follows: At $x = 0$, the surface is not allowed to twist about the curve $z = F(0,y)$, $0 \le y \le b' < d$.

With respect to the boundary C of S, we require that it is twice continuously differentiable with respect to its arclength and that the portion of C intercepted by the sphere of radius d and center a point M of C has a projection on the xy-plane described by the pair of equations.

$$x = f(s), \quad y = g(s), \quad -s_0 \le s \le s_1 \tag{3}$$

where s represents arclength measured from the origin and increasing in the direction of the positive x-axis. The functions f and g possess continuous second derivatives, bounded third ones, and satisfy the conditions,

$$f(0) = g(0) = 0, \quad f'(0) = 1, \quad g'(0) = 0, \quad f'(s)^2 + g'(s)^2 = 1. \tag{4}$$

We note here that the curve described in (3) does not intersect itself since the point (x,y) is in the domain of F.

We call the conditions described by Eqs. (2), (3), and (4) <u>the open surface edge conditions</u> and we make the following definition,

<u>Definition 7</u>. A surface $\bar{S}$ in R^3 is a regular open surface if it is a bounded, simply connected, orientable, smooth open surface, and if it satisfies the open surface edge conditions (2) - (4).

This is the type of surface we will be considering from now on.

C. <u>Description of the problem</u>.

The objective here is to study the behavior of a simple-layer potential and its first derivatives near the boundary of a regular open surface. The potential is of the form

$$U(P) = \int_S \frac{h(M)}{R(P,M)} \, dS \tag{5}$$

where $R(P,M)$ is the Euclidean distance between the points P and M, $M \in S$, $P \notin \bar{S}$, and h is the density function defined on S. As is well known this type of potential has been studied in great detail for S a closed Lyapunov surface, and h Hölder-continuous on it (Günter, 1967; Kellogg, 1953). In the present case the surface is open and the density will be allowed to grow beyond bound in the approach to the boundary. In order to define the density function precisely, we need to introduce a new curvilinear system of coordinates.

With center a point M_0 of C we erect rectangular coordinates, as described in the previous section, and we focus on the portion of Λ' contained in the rectangle $|x| \leq a'$, $0 \leq y \leq b'$. The unit tangent vector at a point of the plane curve (3) is

$$\hat{t}_0 = f'(s)\hat{x} + g'(s)\hat{y}, \tag{6}$$

while the unit normal vector is

$$\hat{n}_0 = \hat{z}\times\hat{t}_0 = -g'(s)\hat{x} + f'(s)\hat{y}. \tag{7}$$

The position vector then to a point (x,y) of Λ' is

$$\hat{x}f(s) + \hat{y}g(s) + \hat{n}_0(s)\rho, \qquad \rho \geq 0 \tag{8}$$

from which we get the transformation

$$x = f(s)-g'(s)\rho, \qquad y = g(s)+f'(s)\rho \tag{9}$$

which will be one-to-one provided the Jacobian

$$\frac{\partial(x,y)}{\partial(s,\rho)} = 1 - \kappa(s) \tag{10}$$

is not zero. Here, $\kappa(s)$ is the curvature of the curve in

(3), i.e.

$$\kappa(s) = f'(s)g''(s) - g'(s)f''(s). \tag{11}$$

By the assumption on f and g, κ is bounded and we can choose ρ sufficiently small so that the Jacobian is strictly positive. We assume that this is the case for the curvilinear rectangle

$$\Lambda = \{(s,\rho) : |s| \leq a, \ 0 \leq \rho \leq b\} \subseteq [-a',a'] \times [0,b']. \tag{12}$$

The density function h is now defined as follows: it is a real-valued function defined over S and satisfying the following two conditions

a) In every closed and connected subset $\overline{S}'$ of $\overline{S}$, bounded away from C, the function h is continuous.

b) In Λ of (12), i.e., near and on the boundary, the function h is of the form

$$h(M) = \frac{\sigma(x,y)}{\rho^{1-\alpha}}, \quad 0 < \alpha \leq 1, \quad \alpha + \mu > 1, \quad (x,y) \in \Lambda \tag{13}$$

where μ is the index introduced in (2) and σ is a function defined on Λ and satisfying the Lipschitz condition

$$|\sigma(M_1) - \sigma(M_2)| \leq A|M_1-M_2|, \quad M_1, \ M_2 \in \Lambda . \tag{14}$$

Definition 8. A real-valued function h defined over a regular open surface $\overline{S}$ is said to be a <u>regular density function</u> for the surface if it satisfies conditions (a) and (b) above.

As mentioned at the beginning of this section, the point P in (5) is near the boundary C of S and is chosen as follows: Given a point M_0 of C, we make it the origin of the rectangular coordinate system described above. We take the point P to be a point on the yz-plane with coordinates (y',z') and such that $0 < (y'^2+z'^2)^{1/2} = \rho' < b$. With this point we also associate polar coordinates (ρ',ϕ') with

$$y' = \rho' \cos \phi', \quad z' = \rho' \sin \phi', \quad \rho' > 0, \quad 0 \leq \phi' \leq 2\pi. \tag{15}$$

In the succeeding sections we will examine the behavior of the line integral of the square of $U(P)$ as well as the square of its spatial derivatives on the circle $y'^2+z'^2 = \rho'^2$ in the limit as $\rho' \to 0$. The results can be summarized in the following two theorems.

<u>Theorem 1</u>: Let $\overline{S} = S \cup C$ be a regular open surface (Def.7) and h a regular density function (Def. 8) defined on S. Let $U(P)$ be the simple-layer potential defined in (5), where $P = (0,y',z')$, $M = (x,y,z)$ with the coordinate system as in the preceeding paragraph. Let also

$$U_1(P) = -\int_S \frac{xh(M)}{R(P,M)^3}\, dS.$$

Then,

$$U(P) = O(1), \quad U_1(P) = O(1), \; \rho' \to 0, \quad 0 \leq \phi' \leq 2\pi.$$

<u>Theorem 2</u>: Let $\bar{S}$ and h be as in Theorem 1 and define

$$U_2(P) = -\int_S \frac{(y-y')h(M)}{R(P,M)^3}\, dS, \quad U_3(P) = -\int_S \frac{(z-z')h(M)}{R(P,M)^3}\, dS.$$

Then,

$$\int_0^{2\pi} \rho' |U_j(P)|^2 d\phi' = \begin{cases} O(\rho'^{2\alpha-1}), & 0 < \alpha < 1 \\ & \qquad\qquad \rho' \to 0 , \\ O(\rho'(\log \rho')^2), & \alpha = 1 \end{cases}$$

where $j = 2,3$.

We remark here that the estimates in Theorem 1 are stronger than those of Theorem 2. Several attempts were made to obtain estimates for U_2 and U_3 like those for U and U_1, but were not successful. In each case the bounds would depend on the angle ϕ' on a way that would make them non-square-integrable with respect to ϕ'.

D. <u>The behavior of the simple-layer</u>

We split the integral in (5) into two integrals by writing

$$U(P) = \int_{S(\Lambda)} \frac{h(M)}{R(P,M)}\, dS + \int_{S-S(\Lambda)} \frac{h(M)}{R(P,M)}\, dS, \qquad (16)$$

where $S(\Lambda)$ is that neighborhood of S about M_0 whose projection on the xy-plane is the region $|s| \leq a$, $0 < \rho \leq b$. The second of these integrals is continuous in a neighborhood of the point M_0, and its limit as $\rho' \to 0$ is equal to the analagous integral obtained when P is replaced by M_0. The first integral can be written as

$$\int_{S(\Lambda)} \frac{h(M)}{R(P,M)}\, dS = \int_{\Lambda} \frac{\sigma(x,y)}{\rho^{1-\alpha}\ (P,M)} \sqrt{1+F_x^{\,2}+F_y^{\,2}}\, dx\, dy$$

$$= \int_{\Lambda} \frac{\sigma^*(s,\rho)}{\rho^{1-\alpha}R(P,M)} \sqrt{1+F_x^{\,2}+F_y^{\,2}}\,(1-\kappa(s)\rho)\,ds\,d\rho, \tag{17}$$

where $R(P,M) = \sqrt{x^2+(y-y')^2+(z-z')^2}$ and $\sigma^*(s,\rho) = \sigma(x(s,\rho),y(s,\rho))$. We then have

$$\left|\int_{S(\Lambda)} \frac{h(M)}{R(P,M)}\, dS\right| \leq M'\int_{\Lambda} \frac{ds\,d\rho}{\rho^{1-\alpha}r'}, \quad r' = \sqrt{x^2+(y-y')^2} \tag{18}$$

where $M' = \max\{|\sigma^*(s,\rho)(1-\kappa(s)\rho)|\sqrt{1+F_x^{\,2}+F_y^{\,2}} : (s,\rho) \in \Lambda\}$. If in (18) we expand x and y about $s = 0$, we find that (see Appendix A),

$$\frac{1}{r'} = \frac{1}{\sqrt{s^2+(\rho-y')^2}}\,(1+0(\varepsilon')), \quad \varepsilon' \to 0^+, \tag{19}$$

provided that $b \leq \varepsilon'/6(|\tilde{\kappa}|+1) < 1$, $a < \varepsilon'/2(1+M_1) < 1$, $|\tilde{\kappa}|b < 1$, with $\tilde{\kappa} = \kappa(0)$, $M_1 = 6M+M^2$, and where M is the maximum of $|f''|$, $|g''|$, $|f'''|$, and $|g'''|$. We can then write

$$\left|\int_{S(\Lambda)} \frac{h(M)}{R(P,M)} dS\right| \leq M(1+0(\varepsilon')) \int_{\Lambda} \frac{\rho^{\alpha-1} ds d\rho}{\sqrt{s^2+(\rho-|y'|)^2}}, \quad \varepsilon' \to 0^+. \tag{20}$$

This last integral exists and is of $0(1)$ as $y' \to 0$, (see Appendix C). We thus have,

$$U(P) = 0(1), \quad \rho' \to 0, \quad 0 \leq \phi' \leq 2\pi \tag{21}$$

and, hence,

$$\int_0^{2\pi} \rho' |U(P)|^2 d\phi' = 0(\rho'), \quad \rho' \to 0. \tag{22}$$

E. The behavior of $U_1(P)$.

In this section we deal with the integral

$$U_1(P) = -\int_S \frac{xh(M)}{R(P,M)^3} dS. \tag{23}$$

As with eq. (16), this integral can be split into two integrals, the second of which is well behaved. For the first one we write,

$$\int_{S(\Lambda)} \frac{xh(M)}{R(P,M)^3} dS = \int_{\Lambda} [\sigma(x,y)\sec\psi - \sigma(0,|y'|)\sec\psi'] \frac{x\rho^{\alpha-1}}{R(P,M)^3} dxdy +$$

$$+ \sigma(0,|y'|)\sec\psi' \int_{\Lambda} \frac{x\rho^{\alpha-1}}{R(P,M)^3} dxdy, \tag{24}$$

where,

$$\sec \psi = \sqrt{1+F_x{}^2+F_y{}^2}\ \big|(x,y), \quad \sec \psi' = \sqrt{1+F_x{}^2+F_y{}^2}\ \big|(0,|y'|), \tag{25}$$

i.e. ψ is the angle the z-axis makes with the normal to the surface at (x,y) while ψ' is the corresponding angle with the normal at $(0,|y'|)$. Since F is twice continuously differentiable we have that

$$|\sec \psi - \sec \psi'| \leq \text{const.}\sqrt{x^2+(y-|y'|)^2}\ ; \tag{26}$$

similarly, since σ satisfies (14)

$$|\sigma(x,y) - \sigma(0,|y'|)| \leq \text{const.}\sqrt{x^2+(y- y'|)^2}\ . \tag{27}$$

Combining the two equations, we obtain

$$|\sigma(x,y)\sec \psi - \sigma(0,|y'|)\sec \psi'| \leq c\ \sqrt{x^2+(y-|y'|)^2}\ , \tag{28}$$

c a constant. We then have that the first integral on the right-hand side of (24) is, in absolute value, less or equal to

$$c\int_\Lambda \frac{|x|\rho^{\alpha-1}\sqrt{x^2+(y-|y'|)^2}}{(x^2+(y-y')^2)^{3/2}}\, dxdy \leq c\int_\Lambda \frac{\rho^{\alpha-1}dxdy}{\sqrt{x^2+(y-|y'|)^2}} \tag{29}$$

This last integral is of the type encountered in (20) and is of $O(1)$ as $y' \to 0$.

For the remaining integral in (24) we write

$$\int_\Lambda \frac{x\rho^{\alpha-1}}{R(P,M)^3}\,dxdy = \int_\Lambda x\rho^{\alpha-1}\left[\frac{1}{R(P,M)^3} - \frac{1}{R^3}\right]dxdy + \int_\Lambda \frac{x\rho^{\alpha-1}}{R^3}\,dxdy, \quad (30)$$

where,

$$R = \sqrt{x^2+(y-y')^2+(F(0,y)-z')^2}\ . \quad (31)$$

For the expression in the brackets we write

$$\frac{1}{R(P,M)^3} - \frac{1}{R^3} = \left(\frac{1}{R(P,M)} - \frac{1}{R}\right)\left(\frac{1}{R(P,M)^2} + \frac{1}{R(P,M)R} + \frac{1}{R^2}\right)\ . \quad (32)$$

For the first term of this expression we have

$$\frac{1}{R(P,M)} - \frac{1}{R} = \frac{R^2-R(P,M)^2}{R(P,M)R[R(P,M)+R]}\ , \quad (33)$$

and

$$R^2 - R(P,M)^2 = [F(0,y)-z']^2 - [F(x,y)-z']^2$$

$$= [F(0,y)-F(x,y)][F(0,y)-z'+F(x,y)-z']$$

so that

$$|R^2 - R(P,M)^2| \le |F(x,y)-F(0,y)|(R+R(P,M))$$

and, by (33),

$$\left|\frac{1}{R(P,M)} - \frac{1}{R}\right| \le \frac{|F(x,y)-F(0,y)|}{R(P,M)R}\ . \quad (34)$$

Combining this result with (2), we have for (32)

$$\left|\frac{1}{R(P,M)^3} - \frac{1}{R^3}\right| \le \frac{A|x|^{1+\mu}}{R(P,M)R}\left[\frac{1}{R(P,M)^2} + \frac{1}{R(P,M)R} + \frac{1}{R^2}\right] \le \frac{3A|x|^{1+\mu}}{r'^4}, \tag{35}$$

where r' is given by (19). The first integral on the right-hand side of (30) is, in absolute value, less than or equal to

$$3A\int_\Lambda \frac{|x|^{2+\mu}\rho^{\alpha-1}}{r'^4}\,dxdy \le 3A\int_\Lambda \frac{\rho^{\alpha-1}dxdy}{\left[\sqrt{x^2+(y-|y'|)^2}\right]^{2-\mu}} \tag{36}$$

The last integral exists and is bounded for all values of y' provided $\alpha + \mu > 1$, (see Appendix C).

In order to evaluate the remaining integral in (30) we resort to the coordinates (s,ρ) introduced in (9). We thus write,

$$\int_\Lambda \frac{x\rho^{\alpha-1}}{R^3}\,dxdy = \int_\Lambda \frac{\rho^{\alpha-1}(f(s)-g'(s)\rho)}{R^3}(1-\kappa(s)\rho)dsd\rho. \tag{37}$$

Expanding the integrand about $s = 0$ (see Appendix B), we have

$$\int_\Lambda \frac{x\rho^{\alpha-1}}{R^3}\,dxdy = (1+0(\varepsilon'))$$

$$\int_\Lambda \frac{\rho^{\alpha-1}[(1-\tilde{\kappa}\rho)s + \frac{1}{2}\tilde{\alpha}s^2]}{[s^2+(\rho-y)')^2+(F(0,\rho)-z')^2]^{3/2}}(1-\kappa(s)\rho)dsd\rho,\ \varepsilon'\to 0^+. \tag{38}$$

From (11)

$$\kappa(s) = \kappa(0) + [f'(s_4)g'''(s_4)-g'(s_4)f'''(s_4)]s,\quad 0 < s_4 < s, \tag{39}$$

so that

$$\int_\Lambda \frac{x\rho^{\alpha-1}}{R^3}\,dxdy = (1+O(\varepsilon'))$$

$$\left\{\int_0^b \int_{-a}^{a} \frac{\rho^{\alpha-1}\tilde{\kappa}(1-\tilde{\kappa}\rho)s}{[s^2+(\rho-y')^2+(F(0,\rho)-z')^2]^{3/2}}\,dsd\rho + O(1)\right\}.$$

Since the integral in the brackets is zero, we have that

$$\int_\Lambda \frac{x\rho^{\alpha-1}}{R^3}\,dxdy = O(1), \quad \rho' \to 0, \quad 0 \le \phi' < 2\pi. \tag{40}$$

Collecting the results from (29), (36), and (40), we have that

$$V_1(P) = O(1), \quad \rho' \to 0, \quad 0 \le \phi' \le 2\pi \tag{41}$$

and, consequently,

$$\int_0^{2\pi} \rho'|V_1(P)|^2 d\phi' = O(\rho'), \quad \rho' \to 0. \tag{42}$$

At this point we have concluded the proof of Theorem 1. In the remaining two sections we will prove Theorem 2.

F. <u>The behavior of</u> $U_2(P)$.

In this section we deal with the integral

$$U_2(P) = -\int_S \frac{(y-y')h(M)}{R(P,M)^3}\,dS. \tag{43}$$

As with (16), this integral can be split into two integrals, the second of which is well behaved. For the first one we write,

$$\int_{S(\Lambda)} \frac{(y-y')h(M)}{R(P,M)^3}\, dS =$$

$$= \int_{\Lambda} [\sigma(x,y)\sec\psi - \sigma(0,|y'|)\sec\psi'] \frac{(y-y')\rho^{\alpha-1}}{R(P,M)^3}\, dxdy +$$

$$+ \sigma(0,|y'|)\sec\psi' \int_{\Lambda} (y-y')\rho^{\alpha-1}\left[\frac{1}{R(P,M)^3} - \frac{1}{R^3}\right] dxdy +$$

$$+ \sigma(0,|y'|)\sec\psi' \int_{\Lambda} \frac{(y-y')\rho^{\alpha-1}}{R^3}\, dxdy. \tag{44}$$

The first two integrals on the righthand side can be treated in the same way as in the previous section. For the last integral we have, according to Appendix B and the last section,

$$\int_{\Lambda} \frac{(y-y')\rho^{\alpha-1}}{R^3}\, dxdy =$$

$$= (1+0(\varepsilon'))\left\{\int_0^b \int_{-a}^{a} \frac{(\rho-y')(1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{[s^2+(\rho-y')^2+(F(0,\rho)-z')^2]^{3/2}}\, dsd\rho + 0(1)\right\} =$$

$$=(1+0(\varepsilon'))\left\{2a\int_0^b \frac{(\rho-y')(1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{[(\rho-y')^2+(F(0,\rho)-z')^2]\sqrt{a^2+(\rho-y')^2+(F(0,\rho)-z')^2}}\, d\rho+0(1)\right\}. \tag{45}$$

If we let $M^*=\max\{|F(0,\rho)| = 0 \leq \rho \leq b\}$, we then have

$$(\rho-y')^2 + (F(0,\rho)-z')^2 \leq (b+|y'|)^2 + (M^*+|z'|)^2.$$

From this inequality together with the requirement that

$$b+|y'| < \frac{a}{\sqrt{2}}, \quad M^*+|z'| < \frac{a}{\sqrt{2}}, \tag{46}$$

we have that

$$(\rho-y')^2 + (F(0,\rho)-z')^2 < a^2 \tag{47}$$

which allows us to expand the radical in (45) into a power series to obtain

$$\int_\Lambda \frac{(y-y')\rho^{\alpha-1}}{R^3}\,dxdy = (1+0(\varepsilon'))\left\{2\int_0^b \frac{(\rho-y')(1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{(\rho-y')^2+(F(0,\rho)-z')^2}d\rho+0(1)\right\}. \tag{48}$$

We observe that the last integral in (48) can be written as

$$\int_0^b \frac{(\rho-y')(1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{(\rho-y')^2+(F(0,\rho)-z')^2}\,d\rho = \mathrm{Re}\int_0^b \frac{\rho^{\alpha-1}(1-\tilde{\kappa}\rho)}{\tau-w}\,d\rho \tag{49}$$

where,

$$\tau = \rho + iF(0,\rho), \quad 0 \le \rho \le b, \quad \text{and} \quad w = y'+iz'. \tag{50}$$

To study the complex integral in (49) we introduce polar coordinates

$$\rho + iF(0,\rho) = re^{i\theta}, \quad \theta = \theta(r) \tag{51}$$

with

$$r = \sqrt{\rho^2+F(0,\rho)^2} \ge \rho. \tag{52}$$

From Appendix D we also have

$$dr > \frac{1}{2}\, d\rho, \qquad r \leq \frac{2}{\sqrt{3}}\, \rho. \tag{53}$$

Taking into consideration that $|\tilde{\kappa}|\rho < 1$, we have that

$$\left|\int_0^b \frac{\rho^{\alpha-1}(1-\tilde{k}\rho)}{\tau-w}\, d\rho\right| \leq \int_0^b \frac{\rho^{\alpha-1}(1+|\tilde{\kappa}|\rho)}{|re^{i\theta}-\rho' e^{i\phi'}|}\, d\rho < \frac{8}{\sqrt{3}} \int_0^{r_b} \frac{r^{\alpha-1}dr}{|re^{i\theta}-\rho' e^{i\phi'}|}\,, \tag{54}$$

where $r_b = r(b)$.

From the discussion up to now we see that

$$|V_2(P)|^2 \leq C_1\left[\int_0^{r_b} \frac{r^{\alpha-1}dr}{|re^{i\theta}-\rho' e^{i\phi'}|}\right]^2 +$$

$$+ 0(1) \int_0^{r_b} \frac{r^{\alpha-1}dr}{|re^{i\theta}-\rho' e^{i\phi'}|} + 0(1), \quad \rho' \to 0, \tag{55}$$

where C_1 is a constant. Since the aim is to estimate the integral

$$\int_0^{2\pi} \rho' |V_2(P)|^2 d\phi' \tag{56}$$

we start by estimating the expression

$$I(\rho') = \int_0^{2\pi} \rho' \left[\int_0^{r_b} \frac{r^{\alpha-1}dr}{\sqrt{r^2+\rho'^2-2r\rho' \cos(\phi'-\theta(r))}}\right]^2 d\phi', \tag{57}$$

which we can write as

$$I(\rho') = \int_0^{2\pi} \rho' d\phi' \left[\int_0^{r_b} \int_0^{r_b} \right. \tag{58}$$

$$\left. \frac{(rq)^{\alpha-1} dr dq}{\sqrt{r^2+\rho'^2-2r\rho' \cos(\phi'-\theta(r))}\sqrt{q^2+\rho'^2-2q\rho' \cos(\phi'-\theta(q))}} \right]$$

We consider the integration on the (r,q,ϕ') space. The integrand is not defined on the surface obtained by revolving the curve $r = \rho'$, $\theta = \phi'$ about the ϕ'-axis; otherwise it is continuous on $(0,r_b] \times (0,r_b] \times [0,2\pi]$. Hence, the integrand is measurable (cf. Sobolev, 1964). We show below that the iterated integral with the order of integration reversed exists. By the Tonelli-Hobson theorem (cf. Apostol, 1974), it is then equal to the original integral. We have that

$$I(\rho') \leq \rho' \int_0^{r_b} \int_0^{r_b} (rq)^{\alpha-1} dr dq \tag{59}$$

$$\left[\int_0^{2\pi} \frac{d\phi'}{r^2+\rho'^2-2r\rho' \cos(\phi'-\theta(r))} \int_0^{2\pi} \frac{d\phi'}{q^2+\rho'^2-2r\rho' \cos(\phi'-\theta(q))} \right]^{1/2},$$

where, above, we used the Cauchy-Schwarz inequality. If we make the substitution $\gamma = \phi' - \theta$ we see that the integrands in the last two integrals are periodic in γ with period 2π. We can then write,

$$I(\rho') \leq \rho' \int_0^{r_b} \int_0^{r_b} (rq)^{\alpha-1} dr dq$$

$$\left[\int_0^{2\pi} \frac{d\phi}{r^2+\rho'^2-2r\rho' \cos \phi'} \int_0^{2\pi} \frac{d\phi}{q^2+\rho'^2-2q\rho' \cos \phi'} \right]^{1/2}. \tag{60}$$

The integrals with respect to ϕ' can be evaluated by using the following integral representation for Legendre functions of the second kind (Magnus et al., 1966, p. 186)

$$Q^{\frac{1}{2}}_{-\frac{1}{2}}(z) = i(2\pi)^{-\frac{1}{2}}(z^2-1)^{\frac{1}{4}} \int_0^\pi (z-\cos t)^{-1}dt. \tag{61}$$

Since (Magnus et al., 1966, p. 172)

$$Q^{\frac{1}{2}}_{-\frac{1}{2}}(z) = i(\tfrac{\pi}{2})^{\frac{1}{2}}(z^2-1)^{-\frac{1}{4}}, \tag{62}$$

we have that

$$\int_0^\pi (z-\cos t)^{-1}dt = \pi(z^2-1)^{-\frac{1}{2}} . \tag{63}$$

From (63) we can compute the integrals in (60):

$$\int_0^{2\pi} \frac{d\phi'}{r^2+\rho'^2-2r\rho' \cos \phi'} = \frac{1}{r\rho'} \int_0^\pi \frac{d\phi'}{\frac{r^2+\rho'^2}{2r\rho'} - \cos \phi'} = \tag{64}$$

$$\frac{\pi}{r\rho'\left[\left(\frac{r^2+\rho'^2}{2r\rho'}\right)^2-1\right]^{\frac{1}{2}}} = \frac{2\pi}{|r^2-\rho'^2|} .$$

Equation (60) then becomes

$$I(\rho') \leq 2\pi\rho' \int_0^{r_b}\int_0^{r_b} \frac{(rq)^{\alpha-1}drdq}{\sqrt{|r^2-\rho'^2|\,|q^2-\rho'^2|}} = 2\pi\rho' \left[\int_0^{r_b} \frac{r^{\alpha-1}dr}{|r^2-\rho'^2|^{\frac{1}{2}}}\right]^2 . \tag{65}$$

We make the change of variables $r = \rho'\xi$ to obtain for (65)

$$I(\rho') \leq 2\pi\rho'^{2\alpha-1}\left[\int_0^{r_b/\rho'} \frac{\xi^{\alpha-1}d\xi}{|\xi^2-1|^{\frac{1}{2}}}\right]^2 . \tag{66}$$

In evaluating (66) we consider the cases $0 < \alpha < 1$ and $\alpha = 1$ separately. For $0 < \alpha < 1$,

$$\int_0^{r_b/\rho'} \frac{\xi^{\alpha-1}d\xi}{|\xi^2-1|^{\frac{1}{2}}} \leq \int_0^{\infty} \frac{\xi^{\alpha-1}d\xi}{|\xi^2-1|^{\frac{1}{2}}} , \tag{67}$$

and the last integral exists as an improper integral. In fact, through simple transformations it can be shown that

$$\begin{aligned}\int_0^{\infty} \frac{\xi^{\alpha-1}d\xi}{|\xi^2-1|^{\frac{1}{2}}} &= \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\alpha/2-1}dt + \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\alpha/2-\frac{1}{2}}\,dt = \\ &= \frac{1}{2}\left[B\left(\frac{1}{2}, \frac{\alpha}{2}\right)+B\left(\frac{1}{2}, \frac{1-\alpha}{2}\right)\right],\end{aligned} \tag{68}$$

where B stands for the Beta function. We then have that

$$I(\rho') \leq \frac{\pi}{2}\,\rho'^{2\alpha-1}\left[B\left(\frac{1}{2}, \frac{\alpha}{2}\right)+B\left(\frac{1}{2}, \frac{1-\alpha}{2}\right)\right]^2, \quad 0 < \alpha < 1. \tag{69}$$

For the case $\alpha = 1$ we compute

$$\begin{aligned}&\int_0^{r_b/\rho'} \frac{d\xi}{|\xi^2-1|^{\frac{1}{2}}} = \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}} + \int_1^{r_b/\rho'} \frac{d\xi}{\sqrt{\xi^2-1}} = \\ &= \frac{\pi}{2} + \log\left[\frac{r_b}{\rho'}\left(1+\sqrt{1-\left(\frac{\rho'}{r_b}\right)^2}\right)\right] \leq \frac{\pi}{2} + 1 + \log\left(\frac{r_b}{\rho'}\right) ;\end{aligned} \tag{70}$$

so that by substitution into (66) we obtain

$$I(\rho') \leq 2\pi\rho'\left[1 + \frac{\pi}{2} + \log\frac{r_b}{\rho'}\right]^2, \quad \alpha = 1 \tag{71}$$

To estimate the second term in (56), i.e. the one resulting from the second term in (55), we use the Cauchy-Schwarz inequality and the results for $I(\rho')$; thus,

$$\int_0^{2\pi} d\phi'\rho' \int_0^{r_b} \frac{r^{\alpha-1}dr}{\sqrt{r^2+\rho'^2-2r\rho'\cos\phi'}} \leq \sqrt{2\pi\rho' I(\rho')} \leq$$

$$\leq \begin{cases} \pi\left[B\left(\frac{1}{2}, \frac{\alpha}{2}\right)+B\left(\frac{1}{2}, \frac{1-\alpha}{2}\right)\right]\rho'^{\alpha}, & 0 < \alpha < 1 \\ 2\pi\rho'\left[1 + \frac{\pi}{2} + \log\left(\frac{r_b}{\rho'}\right)\right], & \alpha = 1. \end{cases} \tag{72}$$

Finally,

$$\int_0^{2\pi} \rho'|V_2(P)|^2 d\phi' = \begin{cases} 0(\rho'^{2\alpha-1}), & 0 < \alpha < 1 \\ 0\left(\rho'\left[\log\left(\frac{r_b}{\rho'}\right)\right]^2\right), & \alpha = 1 \end{cases}, \quad \rho' \to 0. \tag{73}$$

G. <u>The behavior of</u> $U_3(P)$.

In this section we examine the integral

$$U_3(P) = -\int_S \frac{(z-z')h(M)}{R(P,M)^3} dS. \tag{74}$$

As with (16), this integral can be split into two integrals, the second of which is well-behaved. For the first one we write,

$$\int_{S(\Lambda)} \frac{(z-z')h(M)}{R(P,M)^3} dS = \int_{\Lambda} [\sigma(x,y)\sec\psi - \sigma(0,|y'|)\sec\psi'] \frac{(z-z')\rho^{\alpha-1}}{R(P,M)^3} dxdy +$$

$$+\sigma(0,|y'|)\sec\psi' \left\{ \int_{\Lambda} \rho^{\alpha-1}(z-z') \left[\frac{1}{R(P,M)^3} - \frac{1}{R^3} \right] dxdy + \int_{\Lambda} \frac{\rho^{\alpha-1}(z-z')}{R^3} dxdy \right\}. \tag{75}$$

The first two integrals on the right-hand side can be handled as in the previous two sections. For the third one we write

$$\int_{\Lambda} \frac{\rho^{\alpha-1}(z-z')}{R^3} dxdy = \int_{\Lambda} \frac{F(x,y)-F(0,y)}{\rho^{1-\alpha}R^3} dxdy + \int_{\Lambda} \frac{F(0,y)-z'}{\rho^{1-\alpha}R^3} dxdy. \tag{76}$$

By (2), the first of the integrals on the right hand side is, in absolute value, less than or equal to

$$A\int_{\Lambda} \frac{\rho^{\alpha-1}|x|^{1+\mu}}{R^3} dxdy \le A\int_{\Lambda} \frac{\rho^{\alpha-1}\, dxdy}{\left(\sqrt{x^2+(y-|y'|)^2}\right)^{2-\mu}} \quad . \tag{77}$$

The last integral is of the same type as the one in (36). For the last integral in (76) we write, using Appendix B,

$$\int_{\Lambda} \frac{F(0,y)-z'}{\rho^{1-\alpha}R^3} dxdy = (1+0(\varepsilon)) \int_{\Lambda} \frac{\rho^{\alpha-1}[F(0,\rho)-z'+\frac{1}{2}\tilde{\gamma}s^2]}{[s^2+(\rho-y')^2+(F(0,\rho)-z')^2]^{3/2}}$$

$$(1-\kappa(s)\rho)dsd\rho = (1+0(\varepsilon))$$

$$\left\{ \int_0^b \int_{-a}^{a} \frac{[F(0,\rho)-z'](1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{[s^2+(\rho-y')^2+(F(0,\rho)-z')^2]^{3/2}} dsd\rho + 0(1) \right\}. \tag{78}$$

As with (45)-(48), we have

$$\int_{\Lambda} \frac{F(0,y)-z'}{\rho^{1-\alpha}R^3}\, dxdy = (1+O(\varepsilon))\left\{2\int_0^b \frac{[F(0,\rho)-z'](1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{(\rho-y')^2+(F(0,\rho)-z')^2}d\rho+O(1)\right\}. \tag{79}$$

We observe now that

$$\int_0^b \frac{[F(0,\rho)-z'](1-\tilde{\kappa}\rho)\rho^{\alpha-1}}{(\rho-y')^2+(F(0,\rho)-z')^2}\, d\rho = -\mathrm{Im}\int_0^b \frac{\rho^{\alpha-1}(1-\tilde{\kappa}\rho)}{\tau-w}\, d\rho, \tag{80}$$

and, hence, all the results of the previous section can be used here. The sought-after bound is the same as in (73), i.e.

$$\int_0^{2\pi} \rho'|V_3(P)|^2 d\phi' = \begin{cases} O(\rho'^{2\alpha-1}), & 0 < \alpha < 1 \\ O\left(\rho\left[\log\left(\frac{r_b}{\rho'}\right)\right]^2\right), & \alpha = 1 \end{cases} \quad \rho' \to 0. \tag{81}$$

H. Conclusion.

The main results of this work are Theorems 1 and 2 of Section C. In formulating these theorems at the outset we were certain as to the class of density functions we wanted to have included (Def. 8) but not as to the class of open surfaces. Condition (2) on the surface as well as the condition $\alpha + \mu > 1$ in (14) were not anticipated and were made necessary by the manner we proved these theorems. It will be interesting to try to construct alternate proofs which do not employ these two conditions. Another point for investigation, which we also brought up in Section C, is whether order relations with respect to ρ' can be obtained for U_2 and U_3 which would lead to the same result as in Theorem 2. With respect to this last point, we wish to mention that the proof Hayashi (1973) gave for the two-dimensional case is not satisfactory. The reasons for it are outlined below.

Consider a simple, open, and bounded arc L on the yz-plane and make one of the end points the center of a coordinate system as in (15) with (say) the y'-axis tangent to L at the end-point. Consider the function $u(\rho',\phi')$ which is the combination of a simple- and a double-layer potential with densities defined on L. Hayashi shows that $u = 0(1)$ as $\rho' \to 0$. For $\partial u/\partial\rho'$ he, however, reasons as follows: "If we assume that $\partial u/\partial\rho' = 0(\rho'^{\alpha})$, $\rho' \to 0$, then $\alpha > -1$ is necessary in order that u is bounded when $\rho' \to 0$." This statement is not wrong but on the other hand it is not a proof that $\partial u/\partial\rho' = 0(\rho'^{\alpha})$, $\rho' \to 0$, as claimed in his Theorem A.2. What has been proven is that if $u(\rho') = 0(1)$ and $\partial u/\partial\rho' = 0(\rho'^{\alpha})$, then $\alpha > -1$. We wish to make the point here that even though his conclusion is incorrect, this in no way affects the results of his paper for, as in our case, he is interested in an integral of the type

$$\int_0^{2\pi} \rho'|u(\rho',\phi')|d\phi'$$

vanishing in the limit as $\rho' \to 0$ and, we believe, though we have not proven, that this is indeed the case.

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Appendix A. Proof of Equation (20).

We start with Equation (9) which we repeat here for convenience:

$$x = f(s)-g'(s) \quad , \quad y = g(s)+f'(s) \quad . \tag{A.1}$$

Without loss of generality let $s > 0$. Expanding x and y about $s = 0$, we find

$$x = (1-\kappa(0)\rho)s + \frac{1}{2}[f''(s_1)-g'''(s_1)\rho]s^2,$$

$$y = \rho + \frac{1}{2}[g''(s_2)+f'''(s_2)\rho]s^2, \tag{A.2}$$

where $0 < s_1 < s$, $0 < s_2 < s$. We then have

$$x^2+(y-y')^2 = s^2+(\rho-y')^2+(-2\tilde{\kappa} +\tilde{\kappa}^2\rho^2)s^2 +$$

$$\tilde{\beta}(\rho-y')s^2+(1-\tilde{\kappa}\rho)\tilde{\alpha}s^3 + \frac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)s^4, \tag{A.3}$$

where $\tilde{\kappa} = \kappa(0)$, and

$$\tilde{\alpha} = \tilde{\alpha}(s_1,\rho) = f''(s_1)-g'''(s_1)\rho, \quad \tilde{\beta} = \tilde{\beta}(s_2,\rho) = g''(s_2)+f'''(s_2,\rho)\rho. \tag{A.4}$$

Letting B stand for $s^2+(\rho-y')^2$ while A for the rest of the right-hand side of (A.3), we can write

$$\left|\frac{A}{B}\right| \leq |-2\tilde{\kappa}\rho+\tilde{\kappa}^2\rho^2|+|\tilde{\beta}||s|+|1-\tilde{\kappa}\rho||\tilde{\alpha}||s^2 + \frac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)s^2. \tag{A.5}$$

Assuming that f and g have second and third derivatives bounded by the constant M, we get

$$|\tilde{\alpha}| \leq (1+\rho), \quad |\tilde{\beta}| \leq M(1+\rho), \tag{A.6}$$

so that

$$\left|\frac{A}{B}\right| \leq |-2+\tilde{\kappa}\rho||\tilde{\kappa}|\rho+M(1+\rho)|s|+|1-\tilde{\kappa}\rho|M(1+\rho)|s| + \frac{1}{2}M^2(1+\rho)^2|s|^2$$

$$\leq (2+\tilde{\kappa}\rho)|\tilde{\kappa}|\rho+2M(1+\rho)|s|+|\tilde{\kappa}|\rho M(1+\rho)|s| + \frac{1}{2}M^2(1+\rho)^2|s|^2 \tag{A.7}$$

Further requiring that $\rho < 1$, $|\tilde{\kappa}|\rho < 1$, $|s| < 1$, we get for (A.7)

$$\left|\frac{A}{B}\right| < 3|\tilde{\kappa}|\rho+4M|s|+2M|s|+M^3|s| = 3|\tilde{\kappa}|\rho+6M|s|+M^2|s| = 3|\tilde{\kappa}|\rho+M_1|s|, \tag{A.8}$$

where $M_1 = 6M+M^2$. Letting $0 < \varepsilon < 1$ be given, we have that $|A/B| < \varepsilon$ provided that

$$\rho \leq \frac{\varepsilon}{6(|\tilde{\kappa}|+1)} < 1, \quad |s| < \frac{\varepsilon}{2(1+M_1)} < 1, \quad |\tilde{\kappa}|\rho < 1. \tag{A.9}$$

We now examine the expression

$$\frac{1}{\sqrt{B+A}} = \frac{1}{\sqrt{B}\,\sqrt{1+A/B}} = \frac{1}{\sqrt{B}}\left(1 + \frac{1}{\sqrt{1+A/B}} - 1\right) . \tag{A.10}$$

For $0 \leq A/B < \varepsilon$ we have

$$1 - \frac{1}{\sqrt{1+A/B}} < 1 - \frac{1}{\sqrt{1+\varepsilon}} ,$$

while for $-\varepsilon < \frac{A}{B} \leq 0$,

$$\frac{1}{\sqrt{1+A/B}} - 1 < \frac{1}{\sqrt{1-\varepsilon}} - 1 .$$

Since, for $\left|\frac{A}{B}\right| < \varepsilon$,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left| \frac{1}{\sqrt{1+A/B}} - 1 \right| < \frac{1}{2} , \tag{A.11}$$

we have from (A.10)

$$\frac{1}{\sqrt{B+A}} = \frac{1}{\sqrt{B}} [1+0(\varepsilon)], \quad \varepsilon \to 0^+ . \tag{A.12}$$

Without loss of generality we assume that the conditions in (A.9) for (A.12) to hold are satisfied in the curvilinear rectangle Λ of (12).

Appendix B. The expression for $1/R^3$ in terms of the coordinates (s,ρ).

We wish to rewrite $1/R$ of (31) in terms of the coordinates (s,ρ). Let

$$u = F(0,y) = F(0,g(s)+\rho f'(s)) \tag{B.1}$$

Since

$$\left.\frac{\partial y}{\partial s}\right|_{s=0} = g'(0)+\rho f''(0) = 0$$

we have

$$u = F(0,\rho) + \frac{1}{2}\left.\frac{\partial^2 F}{\partial s^2}\right|_{s=s_3} s^2 = F(0,\rho) + \frac{1}{2}\tilde{\gamma}s^2, \quad 0 < s_3 < s \tag{B.2}$$

where

$$\tilde{\gamma} = \tilde{\gamma}(s_3,\rho) = \left.\frac{\partial^2 F(0,y)}{\partial s^2}\right|_{s=s_3}$$

$$= \left.\frac{\partial^2 F(0,y)}{\partial y^2}\right|_{s=s_3}(g'(s_3)+\rho f''(s_3))^2$$

$$+ \left.\frac{\partial F(0,y)}{\partial y}\right|_{s=s_3}(g''(s_3)+\rho f'''(s_3)) \;. \tag{B.3}$$

From (B.2) we have

$$[F(0,y)-z']^2 = [F(0,\rho)-z']^2 + \tilde{\gamma}[F(0,\rho)-z']s^2 + \frac{1}{4}\tilde{\gamma}^2 s^4 \;,$$

so that

$$x^2 + (y-y')^2 + (F(0,y)-z')^2 = B' + A', \tag{B.4}$$

where

$$B' = B + [F(0,\rho)-z']^2 \tag{B.5}$$

$$A' = A + \tilde{\gamma}[F(0,\rho)-z']s^2 + \frac{1}{4}\tilde{\gamma}^2 s^4 \tag{B.6}$$

and A and B have the meaning given to them in Appendix A. Then,

$$\left|\frac{A'}{B'}\right| \le |-2\tilde{\kappa}\rho+\tilde{\kappa}\rho^2|+(|\tilde{\beta}|+|\tilde{\gamma}|)|s|+|1-\tilde{\kappa}\rho||\tilde{\alpha}||s| + \frac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2+\tilde{\gamma}^2)s^2$$

and by (A.5) and (A.8)

$$\left|\frac{A'}{B'}\right| < 3|\tilde{\kappa}|\rho + M_1|s| + |\tilde{\gamma}||s| + \frac{1}{4}\tilde{\gamma}^2 s^2. \tag{B.7}$$

From (B.3) with all derivatives bounded by M,

$$|\tilde{\gamma}| \le M^3(1+\rho)^2+M^2(1+\rho) < (M^3+M^2)(1+\rho)^2 < (M+1)^3(1+\rho)^2 < 4(M+1)^3 \tag{B.8}$$

so that for (B.7) we have

$$\left|\frac{A'}{B'}\right| < 3|\tilde{\kappa}|\rho + [M_1+4(M+1)^3+4(M+1)^6]|s| = 3|\tilde{\kappa}|\rho + M_2|s|. \tag{B.9}$$

Choosing ρ and s according to (A.9) with M_1 substituted by M_2, we have that $|A'/B'| < \varepsilon$.

We now write

$$\frac{1}{R^3} = \frac{1}{(B'+A')^{3/2}} = \frac{1}{B'^{3/2}}\left[1 + \frac{1}{\left(1+\frac{A'}{B'}\right)^{3/2}} - 1\right] . \qquad (B.10)$$

For $0 \le \frac{A'}{B'} < \varepsilon$, we have

$$1 - \frac{1}{\left(1+\frac{A'}{B'}\right)^{3/2}} < 1 - \frac{1}{(1+\varepsilon)^{3/2}}$$

while, for $-\varepsilon < \frac{A'}{B'} \le 0$,

$$\frac{1}{\left(1+\frac{A'}{B'}\right)^{3/2}} - 1 < \frac{1}{(1-\varepsilon)^{3/2}} - 1 \quad .$$

Since, for $|A'/B'| < \varepsilon$,

$$\lim_{\varepsilon \to 0^+} \left| \frac{1}{\left(1+\frac{A'}{B'}\right)^{3/2}} - 1 \right| < 3/2, \qquad (B.11)$$

we have from (B.10)

$$\frac{1}{R^3} = \frac{1}{B'^{3/2}} [1+0(\varepsilon)], \quad \varepsilon \to 0^+ \qquad (B.12)$$

Appendix C. Computation of the integrals in Equations (20) and (36).

The integral in (20) is a special case ($\mu=1$) of the one in (36). As with (18), the integral in (36) can be converted into the following integral in the (s,ρ) plane:

$$\int_{\Lambda} \frac{\rho^{\alpha-1} ds d\rho}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} = \int_0^b \int_{-a}^{a} \frac{\rho^{\alpha-1} ds d\rho}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} =$$

$$= 2\left\{\int_0^{|y'|} \rho^{\alpha-1} d\rho \int_0^a \frac{ds}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} + \right.$$

$$\left. + \int_{|y'|}^{b} \rho^{\alpha-1} d\rho \int_0^a \frac{ds}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}}\right\} =$$

$$= 2|y'|^{\alpha}\left\{\int_0^1 \eta^{\alpha-1} d\eta \int_0^a \frac{ds}{\left(\sqrt{s^2+|y'|^2(1-\eta)^2}\right)^{2-\mu}} + \right.$$

$$\left. + \int_1^{b/|y'|} \eta^{\alpha-1} d\eta \int_0^a \frac{ds}{\left(\sqrt{s^2+|y'|^2(\eta-1)^2}\right)^{2-\mu}}\right\}, \qquad \text{(C.1)}$$

where, above, we made the change of variables $\rho = |y'|\eta$. Letting also $s = |y'|(1-\eta)\xi$ and $s = |y'|(\eta-1)\xi$ in the first and second integrals, respectively, we obtain

$$\int_{\Lambda} \frac{\rho^{\alpha-1} ds\, d\rho}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} =$$

$$= 2|y'|^{\alpha+\mu-1}\left\{\int_0^1 \eta^{\alpha-1}(1-\eta)^{\mu-1} d\eta \int_0^{a/|y'|(1-\eta)} \frac{d\xi}{\left(\sqrt{1+\xi^2}\right)^{2-\mu}} + \right.$$

$$\left. + \int_1^{b/|y'|} \eta^{\alpha-1}(\eta-1)^{\mu-1} d\eta \int_0^{a/|y'|(\eta-1)} \frac{d\xi}{\left(\sqrt{1+\xi^2}\right)^{2-\mu}} \right\}. \qquad \text{(C.2)}$$

We now distinguish two cases.

<u>Case 1</u>: $0 < \tilde{\mu} \leq \mu \leq 1.$

With $0 < \eta < 1$ we have that

$$\int_0^{a/|y'|(1-\eta)} \frac{d\xi}{\left(\sqrt{1+\xi^2}\right)^{2-\mu}} \leq \int_0^{a/|y'|(1-\eta)} \frac{d\xi}{\sqrt{1+\xi^2}} =$$

$$\log\left[\frac{a}{|y'|(1-\eta)} + \sqrt{1 + \frac{a^2}{|y'|^2(1-\eta)^2}}\right] =$$

$$= \log\left(\frac{a}{|y'|(1-\eta)}\right) + \log\left[1 + \sqrt{1 + \frac{|y'|^2(1-\eta)^2}{a^2}}\right] \leq$$

$$\leq \log\left(\frac{a}{|y'|(1-\eta)}\right) + \log\left[1 + \sqrt{1 + \left(\frac{b}{a}\right)^2}\right]. \qquad \text{(C.3)}$$

Similarly, for $1 < \eta \leq b/|y'|$,

$$\int_0^{a/|y'|(\eta-1)} \frac{d\xi}{\left[\sqrt{1+\xi^2}\right]^{2-\mu}} \le \log\left|\frac{a}{|y'|(\eta-1)}\right| + \log\left[1+\sqrt{1+\left[\frac{b}{a}\right]^2}\right] . \tag{C.4}$$

We note here that, since $b > |y'|$ and because of (46), the argument of the first logarithm in each of the last two equations is greater than one.

Substituting (C.3) and (C.4) in (C.2) we obtain

$$\begin{aligned}\int_\Lambda \frac{\rho^{\alpha-1}ds\,d\rho}{\left[\sqrt{s^2+(\rho-|y'|)^2}\right]^{2-\mu}} \le 2|y'|^{\alpha+\mu-1}&\left\{\int_0^1 \eta^{\alpha-1}(1-\eta)^{\mu-1}\log\left(\frac{a}{|y'|(1-\eta)}\right)d\eta\right.\\ &\left.+\int_1^{b/|y'|}\eta^{\alpha-1}(\eta-1)^{\mu-1}\log\left(\frac{a}{|y'|(\eta-1)}\right)d\eta\right\}\\ &+\log\left[1+\sqrt{1+\left(\frac{b}{a}\right)^2}\right]|y'|^{\alpha+\mu-1}\left\{\int_0^1\eta^{\alpha-1}(1-\eta)^{\mu-1}d\eta\,+\right.\\ &\left.+\int_1^{b/|y'|}\eta^{\alpha-1}(\eta-1)^{\mu-1}d\eta\right\} .\end{aligned} \tag{C.5}$$

We now assume that $\alpha+\mu > 1$. For the first integral on the right (call it I_1), we write

$$I_1 \le \int_0^{\frac{1}{2}}\eta^{\alpha+\mu-2}\log\left(\frac{a}{|y'|\,\eta}\right)d\eta + \int_{\frac{1}{2}}^1(1-\eta)^{\alpha+\mu-2}\log\left(\frac{a}{|y'|(1-\eta)}\right)d\eta =$$

$$= \frac{(\frac{1}{2})^{\alpha+\mu-2}}{\alpha+\mu-1}\left[\log\left(\frac{2a}{|y'|}\right)+\frac{1}{\alpha+\mu-1}\right] . \tag{C.6}$$

For the second integral, call it I_2,

$$I_2 \leq \int_1^{b/|y'|} (\eta-1)^{\alpha+\mu-2} \log\left(\frac{a}{|y'|(\eta-1)}\right) d\eta =$$

$$= \frac{(b-|y'|)^{\alpha+\mu-1}}{|y'|^{\alpha+\mu-1}(\alpha+\mu-1)} \left[\log\left(\frac{a}{b-|y'|}\right) + \frac{1}{\alpha+\mu-1}\right] \tag{C.7}$$

The third integral is the Beta-function $B(\alpha,\mu)$, while for the fourth integral, I_4,

$$I_4 \leq \int_1^{b/|y'|} (\eta-1)^{\alpha+\mu-2} d\eta = \frac{(b-|y'|)^{\alpha+\mu-1}}{|y'|^{\alpha+\mu-1}(\alpha+\mu-1)} \quad . \tag{C.8}$$

Equation (C.5) then becomes,

$$\int_\Lambda \frac{\rho^{\alpha-1} ds d\rho}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} \leq \frac{2(\tfrac{1}{2})^{\alpha+\mu-2}}{\alpha+\mu-1} |y'|^{\alpha+\mu-1} \left[\log\left(\frac{2a}{|y'|}\right) + \frac{1}{\alpha+\mu-1}\right] +$$

$$+ \frac{2(b-|y'|)^{\alpha+\mu-1}}{\alpha+\mu-1} \left[\log\left(\frac{a}{b-|y'|}\right) + \frac{1}{\alpha+\mu-1}\right] +$$

$$+ 2\log\left(1+\sqrt{1+\left(\frac{b}{a}\right)^2}\right) \left[B(\alpha,\mu)|y'|^{\alpha+\mu-1} + \frac{(b-|y'|)^{\alpha+\mu-1}}{\alpha+\mu-1}\right], \tag{C.9}$$

an expression which exists for all values of y' .

<u>Case 2</u>: $1 < \mu < 2$

With

$$\mu = 1 + \nu, \quad \nu > 0, \tag{C.10}$$

equation (C.2) becomes

$$\int_{\Lambda} \frac{\rho^{\alpha-1} ds d\rho}{\left(\sqrt{s^2+(\rho-|y'|)^2}\right)^{2-\mu}} =$$

$$= 2|y'|^{\alpha+\nu}\left\{\int_0^1 \eta^{\alpha-1}(1-\eta)^{\nu} d\eta \int_0^{a/|y'|(1-\eta)} \frac{d\xi}{\left(\sqrt{1+\xi^2}\right)^{1-\nu}} + \right.$$

$$\left. \int_1^{b/|y'|} \eta^{\alpha-1}(\eta-1)^{\nu} d\eta \int_0^{a/|y'|(\eta-1)} \frac{d\xi}{\left(\sqrt{1+\xi^2}\right)^{1-\nu}} \right\} \leq$$

$$\leq 2|y'|^{\alpha+\nu} \left\{\int_0^1 \eta^{\alpha-1}(1-\eta)^{\nu} d\eta \int_0^{a/|y'|(1-\eta)} \frac{d\xi}{\xi^{1-\nu}} + \right.$$

$$\left. + \int_1^{b/|y'|} \eta^{\alpha-1}(\eta-1)^{\nu} d\eta \int_0^{a/|y'|(\eta-1)} \frac{d\xi}{\xi^{1-\nu}} \right\} =$$

$$= \frac{2a^{\nu}}{\nu}|y'|^{\alpha} \left\{\int_0^1 \eta^{\alpha-1} d\eta + \int_1^{b/|y'|} \eta^{\alpha-1} d\eta\right\} = \frac{2a^{\nu}b^{\alpha}}{\nu\alpha}. \tag{C.11}$$

We note that for $\mu \geq 2$ the original integral is well-behaved.

Appendix D: Proof of the inequalities in Equation (53).

For the curve $\tilde{C}$ defined by (50) as

$$\tilde{C} = \{\tau = \rho + iF(0,\rho) = 0 \leq \rho \leq b\}$$

we define the position vector $\vec{r}(\rho)$ by means of (51):

$$\vec{r}(\rho) = \rho\hat{\rho} + F(0,\rho)z = r(\cos\theta\ \hat{\rho} + \sin\theta\ \hat{z}), \qquad (D.1)$$

where $\hat{\rho}$ and $\hat{z}$ denote unit vectors in the ρ and z directions, respectively. We can then write for the tangent vector

$$d\vec{r} = (\hat{\rho} + F'\hat{z})d\rho = (\cos\psi\hat{\rho} + \sin\psi\hat{z})\sqrt{1+F'^2}\ d\rho = (\cos\psi\hat{\rho} + \sin\psi\hat{z})ds \qquad (D.2)$$

where s denotes arclength along C, and

$$\cos\psi = \frac{1}{\sqrt{1+F'^2}}, \quad \sin\psi = \frac{F'}{\sqrt{1+F'^2}}, \quad F' = \frac{\partial F(0,\hat{\rho})}{\partial\rho}. \qquad (D.3)$$

On the other hand,

$$dr = \frac{\rho + FF'}{\sqrt{\rho^2+F^2}}d\rho = \frac{\rho + F'F}{\sqrt{\rho^2+F^2}\sqrt{1+F'^2}}ds = (\cos\theta\ \cos\psi + \sin\theta\ \sin\psi)ds \qquad (D.4)$$

For $0 \leq \rho \leq b$ and b sufficiently small we can have

$$\cos\,\theta \geq \frac{\sqrt{3}}{2}\,,\ |\sin\,\theta| \leq \frac{1}{2} \tag{D.5}$$

since $\tilde{C}$ is a Lyapunov curve (cf. Günter, 1967). The same statement can be made about the angle ψ which is the angle the tangent to $\tilde{C}$ at a point makes with the ρ-axis or, equivalently, the normal makes with the z-axis. We then have,

$$dr \geq (\cos\theta\,\cos\psi - \sin|\theta|\sin|\psi|)ds \geq \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2}\right)ds = \frac{1}{2}\,ds \tag{D.6}$$

or

$$dr \geq \frac{1}{2}\,ds > \frac{1}{2}\,d\rho. \tag{D.7}$$

To prove the second of (53) we apply the mean value theorem to the function F:

$$r = \sqrt{\rho^2+F(0,\rho)^2} = \rho\sqrt{1+(F'(0,\rho_1))^2} = \frac{\rho}{\cos\,\psi_1} \leq \frac{2}{\sqrt{3}}\,\rho, \tag{D.8}$$

where $0 < \rho_1 < \rho$.

PART II

The Formulation of the Problem of Scattering of Electromagnetic Waves by an Open, Perfectly Conducting Surface.

Abstract

The problem of scattering of electromagnetic waves by an open, perfectly conducting surface is formulated as a boundary value problem. It is shown that for certain types of open surfaces as well as induced linear current densities the boundary value problem is equivalent to a problem in integral equations of the first kind, and that, moreover, it can have at most one solution.

A. Introduction

One of the most prominent problems of electromagnetic scattering theory is that of scattering of time-harmonic waves by a perfect conductor which occupies a finite region of space and is bounded by a closed surface S. Mathematically, this problem is a boundary value problem stated in terms of Maxwell's equations, radiation conditions, boundary values of the tangential component of the electric field on the surface S, and continuity properties of the scattered fields in space and on the surface. One way of studying this problem, i.e. answering questions on the existence, uniqueness, and properties of solutions, is to convert it to a problem in integral equations where the unknown function is the linear current density on the surface S. This conversion can be accomplished provided the surface and the current density possess certain mathematical properties (cf. Müller, 1969).

A problem very similar to the above but possessing one additional feature is that of scattering of electromagnetic waves by an open, perfectly conducting surface S. The distinctive feature is the very fact that the surface is open. As pointed out in Heins and Silver (1955), it has been known for some time that the conditions imposed on the closed surface problem described above are not sufficient in guaranteeing the uniqueness of the solution of the open surface problem, and that, in order to have uniqueness, additional ones are needed. These conditions vary from author to author (cf. Heins & Silver, 1955; Jones 1964) but, ultimately, all of them have to do with the behavior of the scattered fields near the edge of the open surface and, in effect, demand that no sources of electromagnetic waves are induced on the edge.

Most of the works mentioned in the last two references deal with plane surfaces (or, equivalently, with apertures in perfectly conducting plane screens), and for the most part aim only at obtaining order relations for the scattered fields near the edge of the open surface, quite often at the expense of mathematical rigor. In the present work we propose to formulate the problem of scattering of electromagnetic waves by an open, perfectly conducting surface as a boundary value problem, then convert it to a problem in integral equations, and finally prove that it can have at most one solution. In Section B we present the class of open surfaces that we will consider, some coordinates systems and notation, as well as a brief review of the basic definitions and theorems of vector analysis. In Section C we define the boundary value problem, i.e. present all the conditions that the scattered electromagnetic fields must satisfy. Besides the usual ones required of closed surfaces, as described above, these conditions include statements on the behavior of the scattered fields as well as of the induced current density near the edge of the surface. The scattered fields are required to satisfy an energy condition near the edge which requires that the energy enclosed in a finite region of space is finite and that it vanishes with the volume of the region. The induced current density is required to have a normal component to the edge which achieves a finite limit at the edge, while its component parallel to the edge is allowed to grow beyond bound in the approach to the edge but in a prescribed way. The current density is also required to have first partial derivatives which, near the edge, behave in the manner of its component parallel to the edge. In Section D we assess the physical implications of the requirements on the scattered fields and current density near the edge.

In Sections E and F we prove the equivalence theorem, i.e. that the boundary value problem of Section C is equivalent to a problem in integral equations, while in Section G we prove the uniqueness theorem, i.e. that the problem can have at most one solution. In Section H we offer some concluding remarks, and in Appendices A, B, and C some detailed computations.

B. Preliminary considerations

In this section we will introduce the scattering surface, two coordinate systems associated with it, some notation, as well as define the basic operations of vector analysis.

The surface under consideration is a perfectly conducting surface S bounded by a curve C. We denote by $\overline{S}$ the closure of $S(\bar{S} = S \cup C)$ and we require that $\bar{S}$ is a regular open surface. The definition, properties, and local description of such a surface are given in Part I, Section B, and for this reason we do not repeat them here. In order to reach the objectives of this paper we will have to use this surface in conjunction with the integral theorems of vector analysis. To this end we need to parametrize the curve C with respect to its arclength and to also introduce two coordinate systems.

From an arbitrary point M of C we measure arclength, s', along C and we have the following parametric representation for the curve with respect to an arbitrary rectangular coordinate system xyz:

$$x = \tilde{f}(s'), \quad y = \tilde{g}(s'), \quad z = \tilde{h}(s'), \quad 0 \leq s' \leq L, \tag{1}$$

where L is the length of C. We note that because of the definition of C in Part I, Section B, the functions $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ are twice continuously differentiable in s'; moreover, $\tilde{f}'^2+\tilde{g}'^2+\tilde{h}'^2 = 1$. The unit tangent vector to C is then given by

$$\hat{t} = \tilde{f}(s')' \hat{x}+\tilde{g}(s')' \hat{y}+\tilde{h}(s')' \hat{z} \quad . \tag{2}$$

The unit normal vector to $\bar{S}$ is now chosen so that it is positively oriented with respect to $\hat{t}$. This can be done since $\bar{S}$ is orientable by definition. With respect to the coordinates of (I.1)[1] the unit normal vector is chosen to be

$$\hat{n} = \frac{-F_x\hat{x}-F_y\hat{y}+\hat{z}}{\sqrt{1+F_x^{\,2}+F_y^{\,2}}} \tag{3}$$

i.e. we agree that the local z-axis is positively oriented with respect to $\hat{t}$. For points M of C, we call $\hat{n}$ of (3) the unit normal vector on C. If we also define

$$\hat{\tau} = \hat{n} \times \hat{t} \tag{4}$$

then the triple $(\hat{n},\hat{t},\hat{\tau})$ is a positive triple of orthonormal vectors at each point of C. We note that the vectors $\hat{t}$ and $\hat{\tau}$ lie on the plane tangent to $\overline{S}$ at the point of C under consideration.

The remaining coordinate system is a polar system (ρ',ϕ'), $\rho' \geq 0$, $0 \leq \phi' \leq 2\pi$. It is erected on the plane of the vectors $\hat{\tau}$ and $\hat{n}$ with the pole at the origin of this plane, and the angle ϕ' measured from $\hat{\tau}$ to $\hat{n}$. If $\vec{r}_0$ is the position vector to a point of C, and if $\vec{\rho}\,'$ is the position vector on the $\hat{\tau}$-$\hat{n}$ plane, then the equation

$$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z} = \vec{r}_0+\vec{\rho}\,' \tag{5}$$

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(1) The Roman numeral one (I) refers to Part I of this work.

defines a transformation of points (s',ρ',ϕ') to points (x,y,z). The Jacobian of this transformation is

$$J = \frac{\partial\vec{r}}{\partial s'} \cdot \frac{\partial\vec{r}}{\partial\rho'} \times \frac{\partial\vec{r}}{\partial\phi'} = \rho'\left[1-\rho'\hat{\rho}'\cdot\frac{d\hat{t}}{ds'}\right] . \tag{6}$$

The continuity of the second derivatives of $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ guarantees the boundedness of $|d\hat{t}/ds'|$ which in turn guarantees that, for ρ' sufficiently small, the Jacobian is positive and the transformation one-to-one.

At this point we have completed the discussion of the coordinates systems that will be employed below. Before moving to the subject of vector analysis we introduce some notation that will be used in the following section. We denote by S_+ the side of S facing in the direction of the normal, and by S_- the other side. If f is a function defined in a region containing S, we denote by $f_+(f_-)$ its limit as its argument approaches S from $S_+(S_-)$. Finally, we denote by C_ε the closed curve which is the intersection of S with the cylinder

$$0 \le s' \le L, \quad \rho' = \varepsilon > 0, \quad 0 \le \phi' \le 2\pi, \tag{7}$$

where we take ε small enough so that there is an one-to-one correspondence between points of C_ε and points of C. Due to the smoothness of S, C_ε possesses a tangent vector $\hat{t}$ which we take positively oriented with respect to the normal. If M_ε is a point of C_ε such that $M_\varepsilon \to M \in C$, as $\varepsilon \to 0$, then $\hat{t}(M_\varepsilon) \to \hat{t}(M)$.

We conclude this section by explaining the sense in which the basic operations of vector analysis are to be understood. The ideas that follow have been taken from Müller (1969) who presents them in his book in great detail so that we will present here only what is absolutely necessary for our work. To do so we assume familiarity with the definition of a regular surface and a closed regular surface (Kellogg, 1953; Müller, 1969). A regular region will be a compact point set in R^3 bounded by a closed regular surface.

Definition 1: A sequence of regular regions G_ν is said to converge to a point $\vec{r}_0$ if for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that all G_ν with $\nu \geq N(\varepsilon)$ are entirely within the region $|\vec{r}-\vec{r}_0| \leq \varepsilon$.

Definition 2: Let $\vec{v}(\vec{r})$ be continuous in the neighborhood of the point $\vec{r}_0$. Denote by $||G_\nu||$ the volume of the regular region G_ν and by $F_\nu = \partial G_\nu$, its boundary. If, for each sequence G_ν converging to $\vec{r}_0$, the limit

$$\lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n}\cdot\vec{v}dF$$

exists and is unique, then we set

$$\nabla\cdot\vec{v} = \lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n}\cdot\vec{v}dF,$$

where $\hat{n}$ is the unit normal to F_ν. We call $\nabla\cdot\vec{v}$ the divergence of $\vec{v}$ at $\vec{r}_0$.

Definition 3: With the notation and conditions of Definition 2, if the limit

$$\lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n} \times \vec{v} \, dF$$

exists and is unique for each sequence G_ν converging to $\vec{r}_0$, then we set

$$\nabla \times \vec{v} = \lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n} \times \vec{v} dF$$

We call $\nabla \times \vec{v}$ the curl of $\vec{v}$ at $\vec{r}_0$.

These definitions differ from the usual ones in terms of the operator

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

in that they do not require $\vec{v}$ to have first partial derivatives at $\vec{r}_0$. It can be shown, however, that if $\vec{v}$ has continuous first partials in the neighborhood of $\vec{r}_0$, then the two definitions are equivalent (Müller, 1969). From the definitions above we have the following two theorems (Müller, 1969).

Theorem 1: Let $\vec{v}$ be continuous in the regular region G. Let $\nabla \cdot \vec{v}$ be continuous in each subregion lying entirely in G. If the integral

$$\int_G \nabla \cdot \vec{v} \, dV$$

exists, then

$$\int_F \hat{n}\cdot\vec{v}\, dF = \int_G \nabla\cdot\vec{v}\, dV,$$

where $F = \partial G$.

<u>Theorem</u> 2: Let S be a surface bounded by the curve C so that $\overline{S} = S \cup C$ is a regular open surface. Let $\vec{v}$ and $\nabla\times\vec{v}$ be continuous in a region that contains C. Then

$$\int_S \hat{n}\cdot\nabla\times\vec{v}\, dS = \int_C \hat{t}\cdot\vec{v}\, ds' \ .$$

These two theorems are the well-known *divergence and* Stokes' theorems, respectively. It must be noted that they do not require differentiability of the vector $\vec{v}$. Besides these theorems we will need the surface divergence theorem and for this reason we introduce the following definitions

<u>Definition</u> 4: A sequence of regular open surfaces $\overline{S}_\nu = S_\nu \cup C_\nu$ is said to converge to a point $\vec{r}_0$ if for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that all $\overline{S}_\nu$ with $\nu \geq N(\varepsilon)$ are entirely within the region $|\vec{r}-\vec{r}_0| \leq \varepsilon$.

<u>Definition</u> 5: Let $\vec{v}(\vec{r})$ be continuous in the neighborhood of the point $\vec{r}_0$. Denote by $||\overline{S}_\nu||$ the surface area of the regular open surface $\overline{S}_\nu$. If, for each sequence $\overline{S}_\nu$ converging to $\vec{r}_0$, the limit

$$\lim_{\bar{S}_\nu \to \vec{r}_0} \frac{1}{||\bar{S}_\nu||} \int_{C_\nu} \hat{\tau}\cdot\vec{v}\, ds'$$

exists and is unique, then we set

$$\nabla_0\cdot\vec{v} = -\lim_{\bar{S}_\nu \to \vec{r}_0} \frac{1}{||\bar{S}_\nu||} \int_{C_\nu} \hat{\tau}\cdot\vec{v}\, ds',$$

and we call $\nabla_0\cdot\vec{v}$ the surface divergence of $\vec{v}$ at $\vec{r}_0$.

With this definition we have the following surface divergence theorem (Müller, 1969).

<u>Theorem</u> *3: Let* $\vec{v}$ *and* $\nabla_0\cdot\vec{v}$ *be continuous on the regular open surface* $\bar{S} = S \cup C$. *Then,*

$$\int_S \nabla_0\cdot\vec{v}\, dS = -\int_C \hat{\tau}\cdot\vec{v}\, ds' \quad .$$

We note here that in definition 5 as well as in theorems 2 and 3 the conditions on the surface can be relaxed (cf. Müller, 1969). We close this section with the definition of the gradient of a scalar function.

<u>Definition</u> 6: With the notation of Definition 2, if the scalar function $U(\vec{r})$ is continuous in the neighborhood of the point $\vec{r}_0$, and if

$$\lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n} U dF$$

exists and is unique, then we set

$$\nabla U = \lim_{G_\nu \to \vec{r}_0} \frac{1}{||G_\nu||} \int_{F_\nu} \hat{n}\, U dF,$$

and we call ∇U the gradient of U at $\vec{r}_0$.

C. <u>The open surface problem.</u>

In this section we present in mathematical form the problem of scattering of electromagnetic waves by a perfectly conducting regular open surface $\overline{S} = S \cup C$. Before doing so we introduce a definition that we will need below. In Part I, Section C, we defined a class of functions that we called regular density functions. We now restrict this class to the following.

<u>Definition</u> 7. A real-valued function g defined over a regular open surface $\overline{S}$ is said to be an H-regular density function for the surface if it satisfies Eq. (I.14) and the following condition:

In every closed and connected subset $\overline{S}'$ of S, bounded away from C, the function g is Hölder-continuous, i.e. there exist real numbers B and β such that

$$|g(M_1)-g(M_2)| \leq B|M_1-M_2|^{\beta}, \quad 0 < \beta \leq 1,$$

where B does not depend on the position of M_1 and M_2 in $\overline{S}'$, but may depend on the proximity of $\overline{S}'$ to the boundary C of S.

The open surface problem can now be stated as follows: We wish to find vector-functions $\vec{E}$ and $\vec{H}$ which are defined in R^3-C, are continuous in $R^3-\overline{S}$, also continuous to the surface S from S_+ and S_-, and satisfy the following conditions

(i) Maxwell's equations

$$\nabla\times\vec{E}(\vec{R}) = ikZ\vec{H}(\vec{R}), \quad \nabla\times\vec{H}(\vec{R}) = -ikY\vec{E}(\vec{R}), \quad \vec{R} \in (R^3-\overline{S}) \tag{8}$$

where Z and Y are the free-space impedance and admittance, respectively, so that

$$Z = Y^{-1} = \sqrt{\mu_0/\varepsilon_0}\ , \tag{9}$$

with μ_0 the permeability and ε_0 the permittivity of free space. Equations (8) are also required to hold in the approach to S from S_+ and S_- .

(ii) The Silver-Müller-Wilcox radiation conditions

$$\vec{E} + Z\hat{R}\times\vec{H} = o\left(\frac{1}{R}\right), \quad \vec{H} - Y\hat{R}\times\vec{E} = o\left(\frac{1}{R}\right), \quad R \to \infty \tag{10}$$

uniformly in the non-radial directions. Here, $\vec{R} = R\hat{R}$, with $\hat{R}$ the unit vector in the direction of $\vec{R}$.

(iii) The edge conditions

(a) For $M \in C$ and with respect to the polar coordinate system of Section B

$$\int_0^{2\pi} \vec{E}\cdot\vec{E}^* d\phi' = O(\rho'^{2(\alpha-1)}), \quad \int_0^{2\pi} \vec{H}\cdot\vec{H}^* d\phi' = O(\rho'^{2(\alpha-1)}), \quad \alpha > 0 , \tag{11}$$

where the order is uniform with respect to $M \in C$, and where the asterisk denotes the complex conjugate of the function it is attached to.

(b) On a family of curves $C_\varepsilon \to C$ as described in Section B, the function

$$\psi(M_\varepsilon) = \hat{t}(M_\varepsilon) \cdot (\vec{H}_-(M_\varepsilon)-\vec{H}_+(M_\varepsilon)), \tag{12.a}$$

where $M_\varepsilon \in C_\varepsilon$ *and* $\hat{t}(M_\varepsilon)$ *the unit tangent vector* to C_ε at M_ε , has a finite limit as $\varepsilon \to 0$, and the limit function $\overline{\psi}(M)$, $M \in C$, defined by

$$\overline{\psi}(M) = \lim_{\varepsilon\to 0} \psi(M_\varepsilon), \tag{12.b}$$

with $M_\varepsilon \to M$ as $\varepsilon \to 0$, is continuous at every point M of C. Moreover, the function Ψ defined by

$$\Psi(M_\varepsilon) = \begin{cases} \psi(M_\varepsilon), & 0 < \varepsilon \leq \varepsilon' \\ \overline{\psi}(M), & \varepsilon = 0 \end{cases} \tag{12.c}$$

is an Hölder-continuous function of ε on $0 \leq \varepsilon \leq \varepsilon'$, where ε' is small enough so that there is an one-to-one correspondence between points of C_ε and points of C.

(iv) The boundary condition

$$\hat{n}\times\vec{E}\Big|_{S_+} = \hat{n}\times\vec{E}\Big|_{S_-} = -\hat{n}\times\vec{E}^i\Big|_S \, , \tag{13}$$

where $\vec{E}^i$ is the electric field of a specified plane wave or a dipole source or a combination thereof located off $\bar{S}$. (The condition on the normal component of the magnetic field, i.e.

$$\hat{n}\cdot\vec{H}\Big|_{S_+} = \hat{n}\cdot\vec{H}\Big|_{S_-} = -\hat{n}\cdot\vec{H}\Big|_S^i \tag{14}$$

need not be specified since it can be derived from (13) by taking the divergence of it.)

(v) The density conditions: that the function $\vec{K}(\vec{r}) = \hat{n}(\vec{r}) \times (\vec{H}_-(\vec{r})-\vec{H}_+(\vec{r}))$, $\vec{r} \in S$, has partial derivatives of the first order on S, and that the real and imaginary parts of the components of $\vec{K}$ as well as those of its first partials are H-regular density functions for $\bar{S}$ (Def. 7).

With respect to this statement of the open surface problem we would like to make the following comments. First, the continuity of $\vec{E}$ and $\vec{H}$ in space and in the approach to S from either side together with condition (i) imply that $\nabla\times\vec{E}$ and $\nabla\times\vec{H}$ are continuous in space and in the approach to S from either side. Second, condition (iii.b) is needed only in proving the uniqueness of the solution of the open surface problem but not for converting it into a problem in integral equations. The same is true about the existence of the first partials of $\vec{K}$ in condition (v). Indeed, for the conversion part we will only need that $\vec{K}$ and $\nabla_0 \cdot \vec{K}$ (see Def. 5) have real and imaginary parts which are H-regular density functions for $\overline{S}$. As condition (v) stands $\nabla_0 \cdot \vec{K}$ exists and is equal to the one given in terms of the differential operator ∇ since, by definition 7, the first partials of $\vec{K}$ are Hölder-continuous on S.

The problem described above differs from the corresponding closed surface problem mainly in the addition of conditions (iii). Before undertaking the main task of this paper, i.e. to convert this problem to a problem in integral equations and to also show that it has at most one solution, we would like to first discuss the edge conditions and their physical implications.

D. The edge conditions.

It is well-known that in order to guarantee the uniqueness of solution of the open surface problem we need one more condition than what is required for the closed surface problem (Jones, 1964; Sommerfeld, 1964.b). This condition is associated with the electromagnetic energy enclosed in a finite region of space. If $\vec{e}$ and $\vec{h}$ are electromagnetic fields satisfying Maxwell's equations

$$\nabla \times \vec{e} = -\mu_0 \frac{\partial \vec{h}}{\partial t}, \quad \nabla \times \vec{h} = \varepsilon_0 \frac{\partial \vec{e}}{\partial t}, \tag{15}$$

where ε_0 and μ_0 stand for the permittivity and permeability of free space, respectively, we require that the stored electric and magnetic energy in a finite region of space devoid of sources be finite. For a region V of space these energies are (Sommerfeld, 1964.a)

$$W_e = \frac{\varepsilon_0}{2} \int_V \vec{e} \cdot \vec{e} \, dV, \quad W_m = \frac{\mu_0}{2} \int_V \vec{h} \cdot \vec{h} \, dV. \tag{16}$$

If $\vec{e} = \mathrm{Re}(\vec{E}e^{-i\omega t})$, $\vec{h} = \mathrm{Re}(\vec{H}e^{-i\omega t})$, with $\vec{E}$ and $\vec{H}$ depending on the space variables only, then the time-average electric and magnetic energies are given by

$$\overline{W}_e = \frac{\varepsilon_0}{4} \int_V \vec{E} \cdot \vec{E}^* dV, \quad \overline{W}_m = \frac{\mu_0}{4} \int_V \vec{H} \cdot \vec{H}^* dV, \tag{17}$$

respectively. The bar denotes time-averaging over one period, while the star denotes the complex conjugate of the vector it is attached to (Jones, 1964; Müller, 1969; Stratton , 1941).

In the present case the volume would be a tubular one surrounding the edge C of S. With respect to the coordinate system introduced in Section B, this volume (call it V_ε) is defined by

$$0 \leq s' \leq L, \quad 0 \leq \rho' \leq \varepsilon, \quad 0 \leq \phi' \leq 2\pi \tag{18}$$

The finiteness of energy requirement can then be written as

$$\lim_{\varepsilon \to 0} \int_{V_\varepsilon} \vec{E} \cdot \vec{E}^* dV = 0, \quad \lim_{\varepsilon \to 0} \int_{V_\varepsilon} \vec{H} \cdot \vec{H}^* dV = 0 . \tag{19}$$

Instead of this requirement, we impose the <u>edge conditions</u>

$$\int_0^{2\pi} \vec{E}\cdot\vec{E}\, d\phi' = O(\rho'^{\,2(\alpha-1)}), \quad \int_0^{2\pi} \vec{H}\cdot\vec{H}^* d\phi' = O(\rho'^{\,2(\alpha-1)}), \ \alpha > 0$$

where the order conditions are taken to be uniform with respect to the point M of C. Clearly, if a pair of functions $\vec{E}$ and $\vec{H}$ satisfy the edge conditions, then they also satisfy the energy conditions.

Although conditions (20) are necessary for proving that the open surface problem has at most one solution, they are not sufficient, at least not for the manner in which we approach the question. Condition (iii.b) of Section C is also needed and this we will demonstrate later on. In the meanwhile, we will show that the introduction of such a condition makes sense physically. Let $f(\vec{R})$ be a complex-valued function of $\vec{R}$ defined and with continuous first partials in a bounded domain containing $\overline{S}$ in its interior, and let $\vec{E}$ and $\vec{H}$ be a solution pair for the open surface problem. Let also C_ε represent the closed contour on S which is the intersection of the surface S with the cylinder in (7). By Stokes' theorem we have that

$$\int_{C_\varepsilon} f(\vec{R})\hat{t}\cdot(\vec{H}_- - \vec{H}_+)ds' = -\int_{-C_\varepsilon} f(\vec{R})\hat{t}\cdot\vec{H}_- ds' - \int_{C_\varepsilon} f(\vec{R})\hat{t}\cdot\vec{H}_+ ds' =$$

$$= \int_{\Sigma_\varepsilon} \hat{n}\cdot\nabla\times(f\vec{H})dS = \int_{\Sigma_\varepsilon} \hat{\rho}'\cdot[\nabla f\times\vec{H}+f\nabla\times\vec{H}]dS = -\int_{\Sigma_\varepsilon} [\nabla f\cdot(\hat{\rho}\times\vec{H})+ikYf\hat{\rho}'\cdot\vec{E}]dS, \tag{21}$$

where $-C_\varepsilon$ is transversed in the opposite direction of C_ε, s' denotes arclength, Σ_ε denotes the cylinder in (7), and $\hat{n}$ the unit normal to the cylinder pointing toward the interior.

We can now show that the last integral in (21) vanishes in the limit as $\varepsilon \to 0$. Since $\vec{E}$ and $\vec{H}$ are solutions of the open surface problem they satisfy (20). Using the Cauchy-Schwarz inequality, we have that

$$\left[\iint_{\Sigma_\varepsilon} f\hat{\rho}'\cdot\vec{E}dS\right]^2 \leq \left(\iint_{\Sigma_\varepsilon} |f||\hat{\rho}'\cdot\vec{E}|dS\right)^2 =$$

$$= \left(\int_0^L \int_0^{2\pi} |f||\hat{\rho}'\cdot\vec{E}|\varepsilon\left(1-\varepsilon\hat{\rho}'\cdot \frac{d\hat{t}}{ds'}\right)d\phi' ds'\right)^2 \leq$$

$$\leq \varepsilon^2\left\{\int_0^L \int_0^{2\pi} |f|^2\left(1-\varepsilon\hat{\rho}'\cdot \frac{d\hat{t}}{ds'}\right)^2 d\phi' ds'\right\} \left\{\int_0^L \int_0^{2\pi} |\hat{\rho}'\cdot\vec{E}|^2 d\phi' ds'\right\} \leq$$

$$\leq \varepsilon^2 C_1 \varepsilon^{2(\alpha-1)} = C_1\varepsilon^{2\alpha}, \tag{22}$$

where C_1 is a constant. Similarly,

$$\left|\iint_{\Sigma_\varepsilon} \nabla f \cdot (\hat{\rho}\times\vec{H})\, dS\right| \leq C_2\varepsilon^\alpha$$

where C_2 is a constant. Since $\alpha > 0$, both of these expressions go to zero with ε. We then have that

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(\vec{R})\hat{t} \cdot (\vec{H}_- - \vec{H}_+)ds' = 0. \tag{24}$$

But the integrand in this expression is continuous to the boundary by the edge condition (iii.b) and the assumption on f. This allows us to take the limit before integrating, so that

$$\int_C f(\vec{R})\hat{t} \cdot (\vec{H}_- - \vec{H}_+) ds' = 0 \qquad (25)$$

In order to interpret these last two equations physically we introduce the induced linear current density, $\vec{K}$, on the surface S

$$\vec{K}(\vec{r}) = \hat{n}(\vec{r}) \times (\vec{H}_-(\vec{r}) - \vec{H}_+(\vec{r})), \qquad \vec{r} \in S \qquad (26)$$

a vector tangent to the surface S. Since,

$$\hat{t} \cdot (\vec{H}_- - \vec{H}_+) = (\hat{\tau} \times \hat{n}) \cdot (\vec{H}_- - \vec{H}_+) = \hat{\tau} \cdot n \times (\vec{H}_- - \vec{H}_+) = \hat{\tau} \cdot \vec{K} \quad (27)$$

we have from (24) and (25)

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(\vec{R}) \hat{\tau} \cdot \vec{K} ds' = 0, \quad \int_C f(\vec{R}) \hat{\tau} \cdot \vec{K} ds' = 0. \qquad (28)$$

With $f(\vec{R}) \equiv 1$, and since $\hat{\tau} \cdot \vec{K}$ is normal to C, the second of (28) says that the current (in ampères) entering the surface must be equal to the current leaving it. The first of (28) says basically the same thing. Physically, however, no current is entering or leaving the surface. In fact, we should have that $\hat{\tau} \cdot \vec{K} = 0$ on C. This information is contained in the second of (28) but not in the first. To prove it we use the following theorem (Sobolev, 1964, p. 113).

<u>Theorem</u> 4: If a function g is summable in an open set Ω, and if for any function f which is continuous on Ω the equality $\int_\Omega fg dv = 0$ holds, then g must satisfy the condition $\int_\Omega |g| dv = 0$, and consequently g is equal to zero almost everywhere in Ω. (We remark here that a much stronger version of this theorem can be found in Smirnov (1964, p. 145)).

Identifying g with $\hat{\tau}\cdot\vec{K}$ in the second of (28), we have that $\hat{\tau}\cdot\vec{K}$ is equal to zero almost everywhere on C. Since by (27) and (iii.b) $\hat{\tau}\cdot\vec{K}$ is continuous on C, then $\hat{\tau}\cdot\vec{K} = 0$ everywhere on C. Thus the edge conditions of the open surface problem imply that

$$\hat{\tau}\cdot\vec{K} \equiv 0 \quad \text{on} \quad C. \tag{29}$$

The second of (28) or, equivalently, (25) provides us with another physical statement. Using the surface divergence theorem (theorem 3), we have that

$$0 = \int_C f(R)\hat{\tau}\cdot\vec{K} ds' = -\int_S \nabla_0\cdot(f\vec{K}) dS,$$

or,

$$\int_S \nabla_0\cdot(f\vec{K}) dS = 0. \tag{30}$$

With the accepted definition for the electric charge density, σ, induced on S, i.e.

$$\sigma = \varepsilon_0 \hat{n}\cdot(\vec{E}_- - \vec{E}_+), \tag{31}$$

and the fact that

$$\nabla_0 \cdot \vec{K} = -\hat{n} \cdot \nabla_0 \times (\vec{H}_- - \vec{H}_+) = ikY\hat{n} \cdot (\vec{E}_- - \vec{E}_+) = i\omega\sigma, \tag{32}$$

we have from (30), with $f \equiv 1$, that

$$\int_S \sigma dS = 0, \tag{33}$$

which says that the electric charge induced on S is equal to zero.

E. <u>The integro-differential equation and integral representations</u>.

In this section we convert the open surface problem of Section C into a problem in integral equations.

<u>Theorem 5</u>: If the pair of vector-functions $\{\vec{E}, \vec{H}\}$ is a solution of the open surface problem, then $\vec{E}$ and $\vec{H}$ are given by

$$\vec{E}(\vec{R}') = \int_S \left[\frac{Z}{ik} \nabla_0 \cdot \vec{K}(\vec{R}) \nabla G(\vec{R}|\vec{R}') + ikZG(\vec{R}|\vec{R}')\vec{K}(\vec{R}) \right] dS, \quad \vec{R}' \varepsilon (R^3 - \bar{S}) \tag{34}$$

$$\vec{H}(\vec{R}') = \int_S \vec{K}(\vec{R}) \times \nabla G(\vec{R}|\vec{R}') dS, \quad \vec{R}' \varepsilon (R^3 - \bar{S}) , \tag{35}$$

where,

$$G(\vec{R}|\vec{R}') = - \frac{e^{ik|\vec{R}-\vec{R}'|}}{4\pi|\vec{R}-\vec{R}'|} , \quad \vec{R} \neq \vec{R}' ,$$

and where the linear current density $\vec{K}$ is defined in (26) and satisfies the integro-differential equation

$$-\hat{n}(\vec{r})\times\vec{E}^{i}(\vec{r}) = \int_S \left[\frac{Z}{ik}\nabla_0\cdot\vec{K}\hat{n}(\vec{r})\times\nabla G(\vec{R}|\vec{r})+ikZG(\vec{R}|\vec{r})\hat{n}(\vec{r})\times\vec{K}(\vec{R})\right]dS, \quad \vec{r}\in S. \tag{37}$$

Moreover, on a family of curves $C_\varepsilon \to C$, the function $\hat{\tau}(M_\varepsilon)\cdot K(M_\varepsilon)$, $M_\varepsilon \in C_\varepsilon$, is identical to the function ψ defined in (12.a), and

$$\lim_{\varepsilon\to 0} \hat{\tau}(M_\varepsilon)\cdot\vec{K}(M_\varepsilon) = 0. \tag{38}$$

<u>Proof:</u> *To show that $\vec{E}$ is given by (34) we surround the surface* $\overline{S}$ by a closed surface S_0 which we construct as follows: Let δ and ε be two positive real numbers such that $0 < \delta \leq \varepsilon$. If $\vec{R}_S$ is the position vector describing S, we let

$$S_{+\delta} : \vec{R}_S + \delta\hat{n}\ , \quad S_{-\delta} : \vec{R}_S - \delta\hat{n} \tag{39}$$

Moreover, we let

$$\Sigma_\varepsilon : \vec{r}_0+\vec{\rho}\,', \quad \vec{r}_0 \in C, \quad \rho' = \varepsilon\ . \tag{40}$$

The surface S_0 is the closed surface formed by these three intersecting surfaces as shown in Figure 1.

We apply now Green's second identity in the volume V_0 bounded by the surface S_0, the surface S_R of a sphere whose radius R eventually recedes to infinity, and the surface S' of a sphere of radius r and center at $R' \not\in S_0$. With

$$\overline{\overline{\Gamma}}(\vec{R}|\vec{R}') = ik\nabla\times(G\overline{\overline{I}}),^{(2)} \qquad (41)$$

where G is given by (36), and $\overline{\overline{I}}$ is the identity dyadic, we have that

$$\int_{V_0} [\nabla\times\nabla\times\vec{H}(\vec{R})\cdot\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')-\vec{H}(\vec{R})\cdot\nabla\times\nabla\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')]dV =$$

$$= \int_{S_0+S_R+S'} \hat{n}_0\cdot[\vec{H}(\vec{R})\times\nabla\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')+\nabla\times\vec{H}(\vec{R})\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')]dS^{(3)} \qquad (42)$$

where the unit normal $\hat{n}_0$ to S_0 points away from V_0. Since both $\vec{H}$ and $\overline{\overline{\Gamma}}$ satisfy the equation $\nabla\times\nabla\times(\cdot) - k^2(\cdot) = 0$ in V_0, the volume integral in (42) is zero, while (cf. Asvestas and Kleinman, 1971)

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(2) Double bars over a letter denote a dyadic. All dyadic and vector identities used in this work can be found in Van Bladel (1964).

(3) This identity can be obtained by writing the dyadic in (41) in rectangular components and applying theorem 1. The vector to be used in this theorem is of the form $\vec{v} = \vec{u} \times \vec{w}$. From Muller (1969) we have that if $\vec{u}$, $\vec{w}$, $\nabla\times\vec{u}$, and $\nabla\times\vec{w}$ are continuous in V_0, then

$$\nabla\cdot(\vec{u}\times\vec{w}) = \vec{w}\cdot\nabla\times\vec{u}-\vec{u}\cdot\nabla\times\vec{w} .$$

These conditions are satisfied in (42).

$$\lim_{r\to 0} \int_{S'} \hat{n}_0 \cdot [\vec{H}\times\nabla\times\overline{\overline{\Gamma}}+(\nabla\times\vec{H})\times\overline{\overline{\Gamma}}]dS = -ik\nabla'\times\vec{H}(\vec{R}'). \tag{43}$$

By these last two equations and (8) we have that

$$\vec{E}(\vec{R}') = \frac{Z}{k^2}\int_{S_0+S_R} \hat{n}_0 \cdot [\vec{H}(\vec{R})\times\nabla\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')+\nabla\times\vec{H}(\vec{R})\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')]dS. \tag{44}$$

We next show that the integral over S_R vanishes in the limit as $R \to \infty$. Since,

$$\nabla\times\overline{\overline{\Gamma}} = ik[\nabla\nabla G+k^2G\overline{\overline{I}}] , \tag{45}$$

and

$$\vec{H}\times\nabla\nabla G = (\nabla\times\vec{H})\nabla G-\nabla\times(\vec{H}\nabla G) \tag{46}$$

then, denoting the integral over S_R by I, we have that

$$I = \frac{i}{k}\int_{S_R}\left\{-ik\hat{R}\cdot\vec{E}\nabla G+k^2GZ\hat{R}\times\vec{H}-ik[\hat{R}\cdot\nabla G\vec{E}-\hat{R}\vec{E}\cdot\nabla G]\right\}dS. \tag{47}$$

Employing polar coordinates with center at $\vec{R}'$ and writing

$$\nabla G = -\frac{e^{ikR}}{4\pi}\left(\frac{ik}{R}-\frac{1}{R^2}\right)\hat{R}, \quad dS = R^2 d\Omega \tag{48}$$

we obtain

$$I = -\frac{ik}{4\pi}\int_{S_R} e^{ikR}R(Z\hat{R}\times\vec{H}+\vec{E})d\Omega + \frac{1}{4\pi}\int_{S_R} e^{ikR}\vec{E}d\Omega. \tag{49}$$

The first of these integrals vanishes in the limit as $R \to \infty$

because of (10), while the second vanishes because of a lemma of Wilcox (1956) which in effect states that Maxwell's equation together with the radiation conditions imply that

$$\int_{S_R} |\vec{E}|^2 d\Omega = 0\left(\frac{1}{R^2}\right), \quad R \to \infty. \tag{50}$$

Thus (44) becomes

$$\vec{E}(\vec{R}') = \frac{Z}{k^2}\int_{S_0} \hat{n}\cdot[\vec{H}(\vec{R})\times\nabla\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')+\nabla\times\vec{H}(\vec{R})\times\overline{\overline{\Gamma}}(\vec{R}|\vec{R}')]dS, \tag{51}$$

which by (45) and (46) can be written as

$$\vec{E}(\vec{R}') = \frac{iZ}{k}\int_{S_0} \hat{n}_0\cdot[\vec{H}\times(\nabla\nabla G+k^2G\overline{\overline{I}})-ikY\vec{E}\times\nabla\times(G\overline{\overline{I}})]dS. \tag{52}$$

In order to cast this expression in the usual Stratton-Chu form we use the identities

$$\hat{n}_0\cdot(\vec{H}\times\overline{\overline{I}}) = \hat{n}\times\vec{H} \tag{53}$$

$$\hat{n}_0\cdot[\vec{E}\times\nabla\times(G\overline{\overline{I}})] = \hat{n}_0\cdot[\vec{E}\times(\nabla G\times\overline{\overline{I}})] = (\hat{n}_0\times\vec{E})\cdot(\nabla G\times\overline{\overline{I}}) = (\hat{n}_0\times\vec{E})\times\nabla G \tag{54}$$

$$\begin{aligned}\hat{n}_0\cdot(\vec{H}\times\nabla\nabla G) &= (\hat{n}_0\times\vec{H})\cdot\nabla\nabla G = -(\hat{n}_0\times\vec{H})\cdot\nabla'\nabla G = -\nabla'(\hat{n}_0\vec{H}\cdot\nabla G) = \\ &= \nabla'(\hat{n}_0\cdot\nabla G\times\vec{H}) = \nabla'[\hat{n}_0\cdot\nabla\times(G\vec{H})]-\nabla'(G\hat{n}_0\cdot\nabla\times\vec{H}) = \\ &= \nabla'[\hat{n}_0\cdot\nabla\times(G\vec{H})]-ikY\hat{n}_0\cdot\vec{E}\nabla G\ .\end{aligned} \tag{55}$$

Eq. (52) then becomes

$$\vec{E}(\vec{R}') = \int_{S_0} [\hat{n}_0\cdot\vec{E}\nabla G + ikZG\hat{n}_0\times\vec{H} + (\hat{n}_0\times\vec{E})\times\nabla G]dS + \frac{iZ\nabla'}{k}\int_{S_0}\hat{n}_0\cdot\nabla\times(G\vec{H})dS. \tag{56}$$

The last integral in this expression vanishes by virtue of Stokes' theorem. Letting now $\delta \to 0$, and using the boundary condition (13), we have that

$$\vec{E}(\vec{R}') = \int_{\Sigma_\varepsilon} [\hat{n}_0\cdot\vec{E}\nabla G + ikZG\hat{n}_0\times\vec{H} + (\hat{n}_0\times\vec{E})\times\nabla G]dS +$$

$$+ \int_{S_\varepsilon} [\hat{n}\cdot(\vec{E}_- - \vec{E}_+)\nabla G + ikZG\hat{n}\times(\vec{H}_- - \vec{H}_+)]dS, \tag{57}$$

where S_ε is that portion of S not contained in Σ_ε. The first of these integrals can be shown to vanish as $\varepsilon \to 0$ by means of the edge conditions (11). Since $R' \notin \Sigma_\varepsilon$, G and ∇G are continuous functions of $\vec{R}$ and, hence, bounded. Using the Cauchy-Schwarz inequality as for (22), we have that

$$\left|\int_{\Sigma_\varepsilon} \hat{n}_0\cdot\vec{E}\nabla G dS\right| \le C_1(\vec{R}')\varepsilon^\alpha, \tag{58}$$

$$\left|\int_{\Sigma_\varepsilon} (\hat{n}_0\times\vec{E})\times\nabla G dS\right| \le C_1(\vec{R}')\varepsilon^\alpha \tag{59}$$

and

$$\left|\int_{\Sigma_\varepsilon} G\hat{n}_0\times\vec{H} dS\right| \le C_2(R')\varepsilon^\alpha \tag{60}$$

so that the first integral in (57) is indeed zero in the limit as $\varepsilon \to 0$. The second integral in (57) can be shown to exist as an improper integral by condition (v) of the open surface problem. For $\vec{K} = \hat{n}\cdot(\vec{H}_- - \vec{H}_+)$ is integrable since its components have real and imaginary parts that are H-regular density functions for $\bar{S}$. From (32), $\hat{n}\cdot(\vec{E}_- - \vec{E}_+) = (ikY)^{-1}\nabla_0\cdot\vec{K}$ and, because of the conditions (v) on $\vec{K}$, $\nabla_0\cdot\vec{K}$ exists and its real and imaginary parts are H-regular density functions for $\bar{S}$, hence integrable. Letting

$$\int_S (\ \cdot\ )dS = \lim_{\varepsilon\to 0}\int_{S_\varepsilon}(\ \cdot\ )dS \quad ,$$

we have that

$$\vec{E}(\vec{R}') = \int_S \left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla G(\vec{R}|\vec{R}') + ikZG(\vec{R}|\vec{R}')\vec{K}(\vec{R})\right]dS, \quad R' \in (R^3-\bar{S}). \tag{62}$$

This completes the proof of (34). The proof of (35) follows similar steps and for this reason we omit it.

Equation (38) and the identification of $\hat{\tau}\cdot\vec{K}$ with ψ were proven in Section D so that it only remains to prove (37). This equation can be obtained either from (56) or (62). In (56) the surface S_0 is a closed surface, a fact which enables us to use well-established results regarding the behavior of the integral as $\vec{R}'$ approaches $S_{+\delta}$ or $S_{-\delta}$ (cf. Müller, 1969, Ch. IV). In (62) the surface is open but this equation involves only two terms and for this reason it is the equation we will use to obtain (37). The theorems necessary for the derivation are summarized in Appendix A. From theorem A.3 we have that the second part of the integral in (62) is continuous at every point $\vec{r} \in S$ so that

$$\int_{S_+} \vec{K}(\vec{R})G(\vec{R}|\vec{R}')dS = \int_{S_-} \vec{K}(\vec{R})G(\vec{R}|\vec{R}')dS = \int_S \vec{K}(\vec{R})G(\vec{R}|\vec{r})dS \tag{63}$$

where by $S_+(S_-)$ we mean the limit of the integral as $\vec{R}'$ approaches $\vec{r}$ from the side $S_+(S_-)$. From theorem A.4 we have that the first part of the integral in (62) is continuous at every point $\vec{r}$ of S in the approach to it from either side S_+ or the side S_- and that

$$\int_{S_+(S_-)} \nabla_0\cdot\vec{K}(\vec{R})\nabla G(\vec{R}|\vec{R}')dS =$$

$$= \mp \frac{1}{2}\nabla_0\cdot\vec{K}(\vec{r})\hat{n}(\vec{r}) + \int_S \nabla_0\cdot\vec{K}(\vec{r})\hat{n}\frac{\partial}{\partial n}G(\vec{R}|\vec{r})dS +$$

$$+ \int_S \nabla_0\cdot\vec{K}(\vec{r})\nabla_0 G(\vec{R}|\vec{r})dS + \int_S [\nabla_0\cdot\vec{K}(\vec{R})-\nabla_0\cdot\vec{K}(\vec{r})]\nabla G(\vec{R}|\vec{r})dS, \tag{64}$$

where the minus sign corresponds to the approach from S_+ and the plus from S_-. Combining the last three equations we get

$$\vec{E}_\pm(\vec{r}) = \mp\frac{\hat{n}(\vec{r})}{2}\nabla_0\cdot\vec{K}(\vec{r}) + \int_S\left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla G(\vec{R}|\vec{r})+ikZG(\vec{R}|\vec{r})\vec{K}(\vec{R})\right]dS,\ \vec{r}\in S, \tag{65}$$

where the first part of the integral must be interpreted in the sense of (64). This last equation together with the boundary condition (13) yield (37), and the proof of Theorem 5 is complete.

F. The converse of Theorem 5.

Before we state and prove the converse of Theorem 5 we introduce the following definition

Definition 8. Let $\vec{K}$ be a vector-function defined on S. Then $\vec{K}$ is said to be a surface field on S if $\hat{n}\cdot\vec{K} = 0$ everywhere on S.

Theorem 6. Let $\vec{K}$ be a surface field defined on S and satisfying the following conditions

(i) $\vec{K}$ has partial derivatives of the first order on S, and the real and imaginary parts of the components of $\vec{K}$ as well as those of its first partials are H-regular density functions for $\overline{S}$ (Def. 7).

(ii) On a family of curves $C_\varepsilon \to C$ as described in Section B, the function $\psi(M_\varepsilon) = \hat{\tau}(M_\varepsilon)\cdot\vec{K}(M_\varepsilon)$ has a limit as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \psi(M_\varepsilon) = 0. \tag{66.a}$$

Moreover, the function Ψ defined by

$$\Psi(M_\varepsilon) = \begin{cases} \psi(M_\varepsilon), & 0 < \varepsilon \leq \varepsilon' \\ 0 \quad , & \varepsilon = 0 \quad , \end{cases} \tag{66.b}$$

is an Hölder-continuous function of ε on $0 \leq \varepsilon \leq \varepsilon'$, where ε' is small enough so that there is an one-to-one correspondence between points of C_ε and points of C.

(iii) For all $\vec{r} \varepsilon S$, $\vec{K}$ satisfies the equation

$$-\hat{n}(\vec{r})\times\vec{E}^i(\vec{r}) = \int_S \left[\frac{Z}{ik} \nabla_0\cdot\vec{K}(\vec{R})\hat{n}(\vec{r})\times\nabla G(\vec{R}|\vec{r})+ikZG(\vec{R}|\vec{r})\hat{n}(\vec{r})\times\vec{K}(\vec{R})\right]dS, \tag{67}$$

where $\vec{E}^i$ is the electric field introduced in condition (iv) of the open surface problem.

Then the functions $\vec{E}$ and $\vec{H}$ defined by

$$\vec{E}(\vec{R}') = \int_S \left[\frac{Z}{ik} \nabla_0\cdot\vec{K}(\vec{R})\nabla G(\vec{R}|\vec{R}')+ikZG(\vec{R}|\vec{R}')\vec{K}(\vec{R})\right]dS, \quad \vec{R}' \varepsilon (R^3-\bar{S}) \tag{68}$$

$$\vec{H}(\vec{R}') = \int_S \vec{K}(\vec{R})\times\nabla G(\vec{R}|\vec{R}')dS, \quad \vec{R}' \varepsilon (R^3-S) \tag{69}$$

where G is defined in (36), are solutions of the open surface problem.

To facilitate the proof of this theorem we first introduce the following lemma.

Lemma 1: If the hypotheses of Theorem 6 hold, then

$$\int_S [G(\vec{R}|\vec{R}')\nabla_0\cdot\vec{K}(\vec{R})+\vec{K}(\vec{R})\cdot\nabla_0 G(\vec{R}|\vec{R}')]dS = 0, \quad \vec{R}' \varepsilon (R^3-C). \tag{70}$$

Proof: For $\vec{R}' \varepsilon (R^3-\bar{S})$ we have, by means of the surface divergence theorem (Theorem 3) and (66), that

$$0 = \int_C G(\vec{R}|\vec{R}')\hat{\tau}\cdot\vec{K}(\vec{R})ds' = -\int_S \nabla_0\cdot[G(\vec{R}|\vec{R}')\vec{K}(\vec{R})]dS$$

Expanding the surface integral we obtain (70) for $\vec{R} \in (R^3-\bar{S})$. From Theorem A.3 we have that the first part of the integral in (70) is continuous at every point $\vec{r}\in S$. The same is true for the second part and to prove this we erect a rectangular coordinate system xyz with origin at $\vec{r}$ and the z-axis in the direction of the normal. We can then write

$$\int_S \vec{K}(\vec{R})\cdot\nabla_0 G(\vec{R}|\vec{R}')dS = \int_S \left[\vec{K}_x(\vec{R})\frac{\partial G}{\partial x} + K_y(\vec{R})\frac{\partial G}{\partial y} + K_z(\vec{R})\frac{\partial G}{\partial z}\right]dS =$$

$$= \hat{x}\cdot\int_S K_x(\vec{R})\nabla G dS + \hat{y}\cdot\int_S K_y(\vec{R})\nabla G dS + \hat{z}\cdot\int_S K_z(\vec{R})\nabla G dS. \qquad (71)$$

Applying the results of Theorem A.4, with $\hat{n}(\vec{r}) = \hat{z}$ and $K_z(\vec{r}) = 0$, to this equation we get the continuity property at $\vec{r} \in S$.

We now proceed to prove Theorem 5. We first show that (68) and (69) satisfy Maxwell's equations (8) at all points $\vec{R}' \in (R^3-\bar{S})$:

$$\nabla'\times\vec{E}(\vec{R}') = ikZ\int_S \nabla'\times[G(\vec{R}|\vec{R}')K(\vec{R})]dS =$$

$$= ikZ\int_S \vec{K}(\vec{R})\times\nabla G(\vec{R}|\vec{R}')dS = ikZ\vec{H}(\vec{R}'). \qquad (72)$$

$$\nabla'\times\vec{H}(\vec{R}') = \int_S \nabla'\times[\vec{K}(\vec{R})\times\nabla G(\vec{R}|\vec{R}')]dS =$$

$$= -\int_S [\vec{K}(\vec{R})\nabla^2 G(\vec{R}|\vec{R}') + \vec{K}(\vec{R})\cdot\nabla'\nabla G(\vec{R}|\vec{R}')]dS. \qquad (73)$$

By a standard identity and (70)

$$\int_S \vec{K}(\vec{R})\cdot\nabla'\nabla G(\vec{R}|\vec{R}')dS = \nabla'\int_S \vec{K}(\vec{R})\cdot\nabla G(\vec{R}|\vec{R}')dS = -\nabla'\int_S G(\vec{R}|\vec{R}')\nabla_0\cdot\vec{K}(\vec{R})dS, \tag{74}$$

so that (73) becomes

$$\nabla'\times\vec{H}(\vec{R}') = -\int_S [\nabla_0\cdot\vec{K}(\vec{R})\nabla G(\vec{R}|\vec{R}') - k^2 G(\vec{R}|\vec{R}')\vec{K}(\vec{R})]dS = ikY\vec{E}(\vec{R}'). \tag{75}$$

We note that in (74) we also used the fact that, since $\vec{K}$ is a surface field, $\vec{K}\cdot\nabla G = \vec{K}\cdot\nabla_0 G$.

The functions $\vec{E}$ and $\vec{H}$ are clearly infinitely differentiable at every point $\vec{R}' \in (R^3-\bar{S})$. From (68) and Theorems A.3 and A.4, $\vec{E}$ is continuous to the surface S from S_+ and S_-. From (69) and Theorem A.5 the same is true of $\vec{H}$. Moreover, from (72) and (75) and Theorem A.3 and A.5, we see that Maxwell's equations hold on S in the approach to it from S_+ and S_-.

We next show the radiation conditions (10) are satisfied. Asymptotically,

$$G(\vec{R}|\vec{R}') = \frac{e^{ikR'}}{R'}\,\phi + 0\left(\frac{1}{R'^2}\right), \quad \nabla' G = \frac{e^{ikR'}}{R'}\,(ik\phi\hat{R}') + 0\left(\frac{1}{R'^2}\right), \quad R'\to\infty, \tag{76}$$

where

$$\phi = -\frac{e^{-ik\hat{R}'\cdot\vec{R}}}{4\pi}. \tag{77}$$

Equations (76) are uniformly valid for all $\vec{R} \in S$ and all

$\hat{R}'$. From (68) we have that

$$\vec{E}(\vec{R}') = \frac{Ze^{ikR'}}{R'} \int_S [-\phi\hat{R}'\nabla_0\cdot\vec{K}+ik\phi\vec{K}]dS + 0\left(\frac{1}{R'^2}\right), \tag{78}$$

while from (69)

$$\hat{R}'\times\vec{H}(\vec{R}') = -\frac{e^{ikR'}}{R'} \int_S ik\phi\hat{R}'\times(\vec{K}\times\vec{R}')dS + 0\left(\frac{1}{R'^2}\right) =$$

$$= -\frac{e^{ikR'}}{R'} \int_S ik\phi[\vec{K}-\hat{R}'(\hat{R}'\cdot\vec{K})]dS + 0\left(\frac{1}{R'^2}\right), \tag{79}$$

so that

$$\vec{E}(\vec{R}')+Z\hat{R}'\times\vec{H}(\vec{R}') = \hat{R}' \frac{Ze^{ikR'}}{R'} \int_S [-\phi\nabla_0\cdot\vec{K}+ik\phi\hat{R}'\cdot\vec{K}]dS + 0\left(\frac{1}{R'^2}\right). \tag{80}$$

We now note that

$$\nabla\phi = -\frac{1}{4\pi}\nabla\left(e^{-ik\hat{R}'\cdot\vec{R}}\right) \approx -ik\phi\nabla(\hat{R}'\cdot\vec{R}) = -ik\phi\hat{R}', \tag{81}$$

so that (80) becomes

$$\vec{E}(\vec{R}')+Z\hat{R}'\times\vec{H}(\vec{R}') = -\hat{R}' \frac{Ze^{ikR'}}{R'} \int_S [\phi\nabla_0\cdot\vec{K}+\vec{K}\cdot\nabla\phi]dS + 0\left(\frac{1}{R'^2}\right) =$$

$$= -\hat{R}' \frac{Ze^{ikR'}}{R'} \int_S \nabla_0\cdot(\phi\vec{K})dS + 0\left(\frac{1}{R'^2}\right) = \tag{82}$$

$$= \hat{R}' \frac{Ze^{ikR'}}{R'} \int_C \phi\hat{\tau}\cdot\vec{K}ds + 0\left(\frac{1}{R'^2}\right) = 0\left(\frac{1}{R'^2}\right),$$

where, above, we used the surface divergence theorem, eq. (66), and the fact that $\vec{K}\cdot\nabla\phi = \vec{K}\cdot\nabla_0\phi$. Similarly,

$$\vec{H}(\vec{R}')-Y\hat{R}'\times\vec{E}(\vec{R}') = -\frac{e^{ikR'}}{R'}\int_S ik\phi\vec{K}\times\hat{R}'dS -$$

$$-\frac{e^{ikR'}}{R'}\int_S ik\phi\hat{R}'\times\vec{K}dS + 0\left(\frac{1}{R'^2}\right) = 0\left(\frac{1}{R'^2}\right) \quad . \tag{83}$$

We note that the first integral in (82) is the first term in an asymptotic expansion of (70). Indeed, the radiation conditions can be obtained by first using (70) and then expanding asymptotically. This we do in Appendix B.

We next prove that the edge conditions are satisfied. To this end we let

$$H(\vec{R}|\vec{R}') = G(\vec{R}|\vec{R}') + \frac{1}{4\pi|\vec{R}-\vec{R}'|} = -\frac{1}{4\pi}\left(ik + \frac{(ik)^2}{2}|\vec{R}-\vec{R}'|+\ldots\right). \tag{84}$$

We can then write in place of (68)

$$\vec{E}(\vec{R}') = -\frac{1}{4\pi}\int_S\left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla\frac{1}{|\vec{R}-\vec{R}'|} + \frac{ikZ}{|\vec{R}-\vec{R}'|}\vec{K}(\vec{R})\right]dS +$$

$$+\int_S\left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla H(\vec{R}|\vec{R}')+ikZH(\vec{R}|\vec{R}')\vec{K}(\vec{R})\right]dS, \quad \vec{R}' \in (R^3-\bar{S}), \tag{85}$$

while in place of (69)

$$\vec{H}(\vec{R}') = -\frac{1}{4\pi}\int_S \vec{K}(\vec{R})\times\nabla\frac{1}{|\vec{R}-\vec{R}'|}dS + \int_S\vec{K}(\vec{R})\times\nabla H(\vec{R}|\vec{R}')dS, \quad \vec{R}' \in (R^3-\bar{S}) \; . \tag{86}$$

The second integral in each of these two expressions can be estimated immediately. We first note that

$$H(\vec{R}|\vec{R}') = \frac{1}{4\pi}\left(\frac{1-e^{ik|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|}\right) = -\frac{ik}{4\pi}\sum_{n=0}^{\infty}\frac{(ik|\vec{R}-\vec{R}'|)^n}{(n+1)!}$$

$$\nabla H(\vec{R}|\vec{R}') = \frac{k^2}{4\pi}(\nabla|\vec{R}-\vec{R}'|)\sum_{n=0}^{\infty}\frac{(n+1)(ik|\vec{R}-\vec{R}'|)^n}{(n+2)!}$$

so that

$$|H(\vec{R}|\vec{R}')| \leq \frac{ke^{k|\vec{R}-\vec{R}'|}}{4\pi}, \quad |\nabla H(\vec{R}|\vec{R}')| \leq \frac{k^2 e^{k|\vec{R}-\vec{R}'|}}{4\pi} \tag{87}$$

Since we are verifying the edge conditions the point $\vec{R}'$ is a point near the boundary C of S while $\vec{R}$ is a point of S. Since $\overline{S}$ is bounded, we can draw a sphere of finite diameter D that contains $\overline{S}$ and $\vec{R}'$. Then $|\vec{R}-\vec{R}'| \leq D$, and

$$|H(\vec{R}|\vec{R}')| \leq \frac{ke^{kD}}{4\pi}, \quad |\nabla H(\vec{R}|\vec{R}')| \leq \frac{k^2 e^{kD}}{4\pi}. \tag{88}$$

With these inequalities we have that

$$\left|\int_S\left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla H(\vec{R}|\vec{R}')+ikZH(\vec{R}|\vec{R}')\vec{K}(\vec{R})\right]dS\right| \leq$$

$$\leq \frac{Zke^{kD}}{4\pi}\int_S(|\nabla_0\cdot\vec{K}(\vec{R})|+k|\vec{K}(\vec{R})|)dS, \tag{89}$$

$$\left|\int_S\vec{K}(\vec{R})\times\nabla H(\vec{R}|\vec{R}')dS\right| \leq \int_S|\vec{K}(\vec{R})||\nabla H(\vec{R}|\vec{R}')|dS \leq \frac{k^2 e^{kD}}{4\pi}\int_S|\vec{K}(\vec{R})|dS. \tag{90}$$

Since $|\vec{K}|$ and $|\nabla_0 \cdot \vec{K}|$ are integrable by assumption, we see that the integrals involving $H(\vec{R}|\vec{R}')$ in (85) and (86) are bounded, so that in verifying (11) we need concern ourselves only with the remaining integrals in these two equations. That integrals of this type do satisfy the conditions in (11) is the subject of Part I of this work.

To show that the edge condition (iii.b) is also satisfied we use (69) and Theorem A.5 to get that at every point $\vec{r}$ of S

$$\vec{H}_-(\vec{r})-\vec{H}_+(\vec{r}) = -\hat{n}(\vec{r})\times\vec{K}(\vec{r}). \tag{91}$$

On any of the curves C_ε,

$$\hat{t}(\vec{r})\cdot(\vec{H}_-(\vec{r})-\vec{H}_+(\vec{r})) = -\vec{K}(\vec{r})\cdot\hat{t}(\vec{r})\times\hat{n}(\vec{r}) = \hat{\tau}(\vec{r})\cdot\vec{K}(\vec{r}), \tag{92}$$

By condition (ii) in the present theorem we see that the limit as $\varepsilon \to 0$ exists and is equal to zero, so that the edge condition (iii.b) is indeed satisfied.

The boundary condition (13) can be shown to be satisfied by first evaluating (68) at a point $\vec{r}$ of S. Using (A.13) and (A.14) we have that

$$\hat{n}\times\vec{E}\Big|_{S_+} = \hat{n}\times\vec{E}\Big|_{S_-} = \int_S \left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\hat{n}(\vec{r})\times\nabla G(\vec{R}|\vec{r})+ikZG(\vec{R}|\vec{r})\hat{n}(\vec{r})\times\vec{K}(\vec{R})\right]dS.$$

Combining this with (67) we obtain (13).

Finally, from (91) we have that

$$\hat{n}(\vec{r}) \times (\vec{H}_-(\vec{r})-\vec{H}_+(\vec{r})) = \vec{K}(\vec{r}), \quad \vec{r} \in S \tag{94}$$

so that, with condition (i) above, the density conditions (v) of the open surface problem are satisfied. This completes the proof of Theorem 6 and we can now state the equivalence theorem.

Theorem 7: The pair of vector-functions $\{\vec{E},\vec{H}\}$ is a solution of the open surface problem if, and only if, $\vec{E}$ is given by (68) and $\vec{H}$ by (69), where $\vec{K}$ satisfies conditions (i)-(iii) of Theorem 6.

G. The uniqueness of the solution of the open surface problem

In this section we prove that the open surface problem has at most one solution. To this end we assume that there exist two solution pairs $\{\vec{E}_1,\vec{H}_1\}$ and $\{\vec{E}_2,\vec{H}_2\}$ and we form their difference

$$\vec{E} = \vec{E}_1-\vec{E}_2, \qquad \vec{H} = \vec{H}_1-\vec{H}_2 . \tag{95}$$

The pair $\{\vec{E},\vec{H}\}$ is a solution of the open surface problem with the boundary condition (13) replaced by

$$\hat{n}\times\vec{E}\Big|_{S_+} = \hat{n}\times\vec{E}\Big|_{S_-} = 0. \tag{96}$$

We now apply the divergence theorem to the function $\vec{E}^{*}\times\vec{H}$ in the region bounded by the surface S_0 defined through (39) and (40) (see also Figure 1) and the surface S_R of a sphere whose radius R eventually recedes to infinity. Calling this region V_0 we have that

$$\int_{V_0} \nabla\cdot(\vec{E}^{*}\times\vec{H})\,dV = \int_{S_0+S_R} \hat{n}\cdot(\vec{E}^{*}\times\vec{H})\,dS. \tag{97}$$

The surface integrals over $S_{+\delta}$ and $S_{-\delta}$ vanish in the limit as $\delta \to 0$ because of (96). The surface integral over Σ_ε vanishes in the limit as $\varepsilon \to 0$ as shown in Appendix C. Thus the integral over S_0 vanishes in the limit as S_0 collapses to S. For the integral over S_R we use the radiation conditions (10) to rewrite the integrand as

$$\hat{n}\cdot(\vec{E}^{*}\times\vec{H}) = -\vec{E}^{*}\cdot(\hat{n}\times\vec{H}) = -\vec{E}^{*}\cdot(\vec{R}\times\vec{H}) = Y\vec{E}\cdot\vec{E}^{*} + o\left(\frac{1}{R}\right)\vec{E}^{*} = Y\vec{E}\cdot\vec{E}^{*} + o\left(\frac{1}{R^2}\right), \tag{98}$$

since $\vec{E} = 0(1/R)$ (Wilcox, 1956). Since

$$\nabla\cdot(\vec{E}^{*}\times\vec{H}) = \vec{H}\cdot\nabla\times\vec{E}^{*}-\vec{E}^{*}\cdot\nabla\times\vec{H} = -ikZ\vec{H}\cdot\vec{H}^{*}+ikY\vec{E}\cdot\vec{E}^{*} ,$$

we have for (97)

$$ik\int_V (Y\vec{E}\cdot\vec{E}^{*}-Z\vec{H}\cdot\vec{H}^{*})\,dV = Y\int_{S_R} \vec{E}\cdot\vec{E}^{*}\,dS + 0(1), \quad R \to \infty \tag{99}$$

where $V = \lim\limits_{\varepsilon\to 0}\left(\lim\limits_{\delta\to 0} V_0\right)$. Since the left-hand side of this expression is purely imaginary, we have that

$$\lim_{R\to\infty} \int_{S_R} \vec{E}\cdot\vec{E}^* dS = 0. \tag{100}$$

For points not on $\overline{S}$, $\vec{E}$ satisfies the vector wave equation $\nabla\times\nabla\times\vec{E}-k^2\vec{E} = 0$. From the second of (8), definition (2), and Stokes' theorem, we have that $\nabla\cdot\vec{E} = 0$ so that

$$\nabla^2\vec{E}+k^2\vec{E} = 0 . \tag{101}$$

Eqs. (100) and (101) imply that outside a sphere of radius R enclosing $\overline{S}$, the electric field vanishes identically. This is due to the following result by F. Rellich (Müller, 1959).

Theorem 9: If $V(\vec{R})$ is a solution of the Helmholtz equation

$$\nabla^2 V+k^2 V = 0, \quad \text{Re}k > 0$$

for $R = |\vec{R}| \geq C$, and if

$$\int_\Omega |V(\vec{R})|^2 d\Omega = o\left(\frac{1}{R^2}\right), \quad R \to \infty,$$

then $V(\vec{R})$ vanishes identically in $|\vec{R}| \geq C$

Thus $\vec{E}$ and $\vec{H}$ vanish identically exterior to the sphere. Between the sphere and $\overline{S}$, they also vanish being analytic solutions of (101).

We have then shown that $\vec{E}_1 \equiv \vec{E}_2$ and $\vec{H}_1 \equiv \vec{H}_2$ for all points $\vec{R} \not\in \overline{S}$. Since these fields are continuous to the surface S from S_+ and S_-, these relations hold also for points $\vec{R}$ of S. We can thus state the following theorem:

<u>Theorem</u> 8: The open surface problem has at most one solution.

H. <u>Concluding remarks</u>.

The main results of this work are the equivalence and uniqueness theorems (Theorems 7 and 8, respectively.) The equivalence theorem is an immediate consequence of Theorem 5 and 6, while the proof of the uniqueness theorem rests heavily upon the results of Appendix C. In proving Theorems 5 and 6 we note that condition (iii.b) in the statement of the open surface problem (Section C) was not needed at all. What was really needed was the statement in (24) which is a direct consequence of (11). The statement in (29), which follows from (24), (25), and condition (iii.b), is sufficient for the proofs but not necessary. Moreover, in deriving it, we only needed that the function in (12.c) be continuous but not Hölder-continuous. The same is true regarding the existence of the first partials of $\vec{K}$ in condition (v), i.e. in proving Theorems 5 and 6 we only needed the condition that the surface divergence of $\vec{K}$ exists and that its real and imaginary parts are H-regular density functions for $\overline{S}$.

The "excess" conditions then were necessary only in proving the uniqueness theorem, and specifically in showing that the integral in (C.1), Appendix C, vanishes with ε. The question that immediately arises here is whether we can employ the divergence theorem and write

$$\int_{\Sigma_\varepsilon} \hat{n}\cdot(\vec{E}^\star\times\vec{H})\,dS = -\int_{V_\varepsilon} \nabla\cdot(\vec{E}^\star\times\vec{H})\,dV = ik\int_{V_\varepsilon} (Z\vec{H}\cdot\vec{H}^\star - Y\vec{E}\cdot\vec{E}^\star)\,dV,$$

where V_ε is the region enclosed by Σ_ε. The last integral here would then vanish with ε because of (11). Because V_ε includes the edge C, where the functions are not defined on it, we could not answer this question for sure, and for this reason we followed the longer proof of Appendix C, and in the process we had to impose the additional conditions.

Finally, in the last steps of the proof of Appendix C (eqs. (C.14)-(C.21)), we note that condition (I.2) on the open surface plays an indispensible role, which once more raises the question of how fundamental this condition is for problems of scattering by open surfaces.

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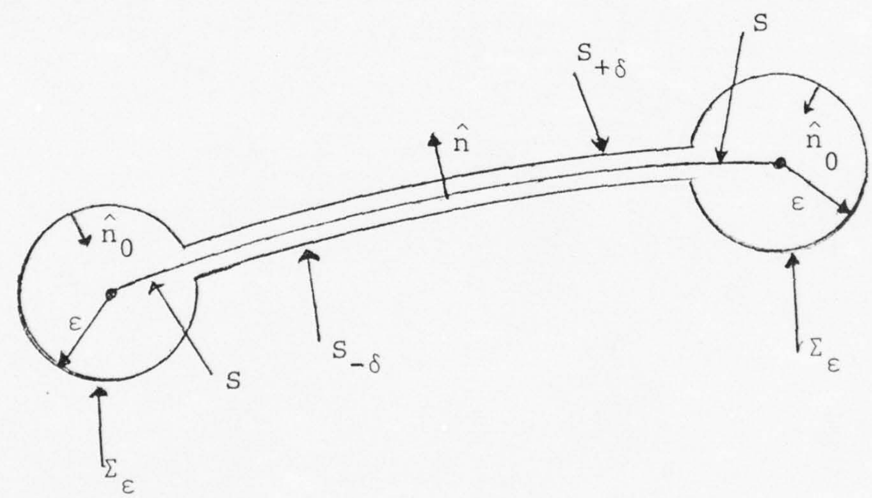

Figure 1. A Cross-Section of the Surfaces S and S_0 .

Appendix A. The behavior of certain integrals.

In this appendix we discuss the continuity properties of integrals of the type

$$U(\vec{R}') = \int_S \frac{h(\vec{R})}{|\vec{R}-\vec{R}'|}\, dS \quad \text{and} \quad \vec{V}(\vec{R}') = \int_S g(\vec{R})\nabla \frac{1}{|\vec{R}-\vec{R}'|}\, dS,$$

where the function h is a regular density function (Part I, Def. 8) for $\bar{S}$, while g is an H-regular density function (Def. 7) for $\bar{S}$. It is clear that U and $\vec{V}$ are infinitely differentiable functions at all points $\vec{R}'$ of space which do not belong to $\bar{S}$. The only other points of interest are those points $\vec{r}$ which belong to S (but not to its boundary C). The proofs of the continuity properties of U and $\vec{V}$ at such points closely follow those for the same type of integrals over closed surfaces. For this reason we will present here only the results and will refer the reader to Müller (1969) for the proofs.

Theorem A.1: If h is a regular density function for $\bar{S}$, then the function

$$U(R') = \int_S \frac{h(\vec{R})}{|\vec{R}-\vec{R}'|}\, dS, \qquad (A.1)$$

is continuous at all points $\vec{r} \in S$.

We outline part of the proof of this theorem to point out how it differs from the one for closed surfaces. Let $\vec{r} \in S$.

Since $\overline{S}$ is a regular open surface, we can erect a rectangular coordinate system with origin at $\vec{r}$ and the z-axis along the normal $\hat{n}(\vec{r})$ to S at $\vec{r}$. Portion of S can be described in terms of this coordinate system in the manner of (I.1). Let d be a positive number such that the sphere $|\vec{R}'-\vec{r}| \leq d$, and with center at $\vec{r}'$, contains only that part of S that is described as in (I.1) and, if necessary, we restrict d so that no points of the boundary C are contained in the sphere. We denote this portion of S by $S_d(\vec{r})$ and let

$$U(\vec{R}') = \int_{S-S_d(\vec{r})} \frac{h(\vec{R})}{|\vec{R}-\vec{R}'|}\, dS + \int_{S_d(\vec{r})} \frac{h(\vec{R})}{|\vec{R}-\vec{R}'|}\, dS = U_1(\vec{R}')+U_2(\vec{R}'). \tag{A.2}$$

We note that U_1 exists as an improper integral as far as the integration in the neighborhood of the boundary C is concerned because of the properties of h. As for the singularity at $\vec{r}$, Müller (1969, lemma 66) has shown that $U_1(\vec{R}') = 0(d)$. Thus U_1 exists as an improper integral if $\vec{R}' \in S$. For $|\vec{R}'-\vec{r}| \leq d/2$ and $|\vec{R}-\vec{r}| \geq d$, we have that

$$\left|\frac{1}{|\vec{R}-\vec{R}'|} - \frac{1}{|\vec{R}-\vec{r}|}\right| \leq \frac{|\vec{R}'-\vec{r}|}{|\vec{R}-\vec{R}'|\,|\vec{R}-\vec{r}|} \leq \frac{2}{d^2}\,|\vec{R}'-\vec{r}|, \tag{A.3}$$

so that

$$|U_1(\vec{R}')-U_1(\vec{r})| \leq \frac{2}{d^2}\,|\vec{R}'-\vec{r}| \int_{S-S_d(\vec{r})} |h(\vec{R})|\,dS. \tag{A.4}$$

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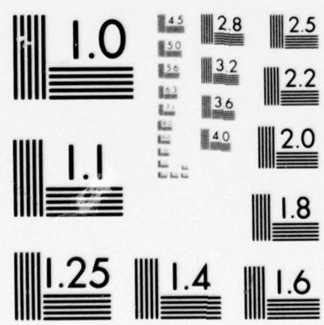

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This last integral exists by virtue of the absolute integrability of h. Thus the continuity of U_1 at $\vec{r}$ has been proven. The proof for the continuity of U_2 is given in Müller (1969, lemma 69). Indeed, the proofs in Muller and the ones required here differ mostly with respect to the integral over $S-S_d(\vec{r})$. In [illegible]er the integrand is continuous, and hence bounded, while here it could become unbounded near the boundary C but it is still integrable.

<u>Lemma A.1</u>: The function

$$\vec{W}(\vec{R}') = \int_S \nabla \frac{1}{|\vec{R}-\vec{R}'|} \, dS \tag{A.5}$$

is continuous at every point $\vec{r} \in S$ in the approach to it from either the side S_+ or the side S_-. Moreover,

$$\int_{S_+} \nabla \frac{1}{|\vec{R}-\vec{R}'|} \, dS = 2\pi\hat{n}(\vec{r}) + \int_S \hat{n} \frac{\partial}{\partial n} \frac{1}{|\vec{R}-\vec{r}|} \, dS + \int_S \nabla_0 \frac{1}{|\vec{R}-\vec{r}|} \, dS, \tag{A.6}$$

$$\int_{S_-} \nabla \frac{1}{|\vec{R}-\vec{R}'|} \, dS = -2\pi\hat{n}(\vec{r}) + \int_S \hat{n} \frac{\partial}{\partial n} \frac{1}{|\vec{R}-\vec{r}|} \, dS + \int_S \nabla_0 \frac{1}{|\vec{R}-\vec{r}|} \, dS, \tag{A.7}$$

where by $S_+(S_-)$ we mean the limit of the integral as $\vec{R}'$ approaches $\vec{r}$ from the side $S_+(S_-)$, and where ∇_0 denotes the surface gradient, i.e.

$$\nabla = \nabla_0 + \hat{n} \frac{\partial}{\partial n} \quad . \tag{A.8}$$

Proof: Let $\bar{T}$ be a regular open surface having the same boundary C as $\bar{S}$ but no other points in common and such that $\bar{T} \cup \bar{S}$ (which is a smooth closed surface) has an exterior normal which coincides with that on $\bar{S}$ at their common points. According to Müller (1969, Lemma 70), lemma A.1 is true for the closed surface $\bar{T} \cup \bar{S}$ and, hence, for S.

Remark: In lemma 70 of Müller it is also proven that the last integral in (A.6) is continuous at all points $\vec{r} \in (R^3 - C)$.

Following Müller (1969, Theorems 41 and 42) we now have the following result

Theorem A.2: The function

$$\vec{V}(\vec{R}') = \int_S g(\vec{R}) \nabla \frac{1}{|\vec{R}-\vec{R}'|} \, dS, \tag{A.9}$$

where g is an H-regular density function for the surface $\bar{S}$, is continuous at every point $\vec{r} \in S$ in the approach to it from either the side S_+ or the side S_-. Moreover,

$$\int_{S_+(S_-)} g(\vec{R}) \nabla \frac{1}{|\vec{R}-\vec{R}'|} \, dS = \pm 2\pi g(\vec{r})\hat{n}(\vec{r}) + \int_S g(\vec{r})\hat{n} \frac{\partial}{\partial n} \frac{1}{|\vec{R}-\vec{r}|} \, dS +$$

$$+ \int_S g(\vec{r}) \nabla_0 \frac{1}{|\vec{R}-\vec{r}|} \, dS + \int_S [g(\vec{R}) - g(\vec{r})] \nabla \frac{1}{|\vec{R}-\vec{r}|} \, dS , \tag{A.10}$$

where the plus sign corresponds to the approach from S_+, and the minus from S_-.

Moreover, if we let

$$H(\vec{R}|\vec{R}') = G(\vec{R}|\vec{R}') + \frac{1}{4\pi|\vec{R}-\vec{R}'|} = -\frac{1}{4\pi}\left(ik + \frac{(ik)^2}{2}|\vec{R}-\vec{R}'|+\ldots\right), \quad (A.11)$$

then, according to Müller (1969, lemmas 73 and 74), the functions

$$\int_S h(\vec{R})H(\vec{R}|\vec{r})dS \quad \text{and} \quad \int_S g(\vec{R})\nabla H(\vec{R}|\vec{r})dS$$

are continuous at all points $\vec{r}$ of S, so that together with Theorems A.1 and A.2 we have the following results.

Theorem A.3: If h is a regular density function for the surface $\bar{S}$, then the function

$$u(\vec{R}') = \int_S h(\vec{R})G(\vec{R}|\vec{R}')dS \quad (A.12)$$

is continuous at every point $\vec{r}$ of S.

Theorem A.4: If g is an H-regular density function for the surface $\bar{S}$, then the function

$$\vec{v}(\vec{R}') = \int_S g(\vec{R})\nabla G(\vec{R}|\vec{R}')dS \quad (A.13)$$

is continuous at every point $\vec{r}$ of S in the approach to it from either the side S_+ or the side S_-. Moreover,

$$\int_{S_+(S_-)} g(\vec{R})\nabla G(\vec{R}|\vec{R}')\,dS = \mp \frac{1}{2}\, g(\vec{r})\hat{n}(\vec{r}) +$$

$$+ \int_S g(\vec{r})\hat{n}\,\frac{\partial}{\partial n}\, G(\vec{R}|\vec{r})\,dS + \int_S g(\vec{r})\nabla_0 G(\vec{R}|\vec{r})\,dS +$$

$$+ \int_S [g(\vec{R})-g(\vec{r})]\nabla G(\vec{R}|\vec{r})\,dS, \tag{A.14}$$

where the minus sign corresponds to the approach from S_+, and the plus from S_-.

Finally we examine functions of the type

$$\int_S \vec{g}(\vec{R})\times\nabla G(\vec{R}|\vec{R}')\,dS,$$

where the vector-function $\vec{g}$ has components which are H-regular density functions for $\overline{S}$. To this end, and with $\vec{R}' \not\in \overline{S}$, we write

$$\int_S \vec{g}(\vec{R})\times\nabla\frac{1}{|\vec{R}-\vec{R}'|}\,dS = \int_S [\vec{g}(\vec{R})-\vec{g}(\vec{r})]\times\nabla\frac{1}{|\vec{R}-\vec{R}'|}\,dS+\vec{g}(\vec{r})\times\int_S \nabla\frac{1}{|\vec{R}-\vec{R}'|}\,dS. \tag{A.15}$$

The first integral is continuous at all points $\vec{R}' \in (R^3-C)$ and the proof is as in Theorem 41 in Müller (1969). For the second integral we employ Lemma A.1 so that

$$\int_{S_+(S_-)} \vec{g}(\vec{R})\times\nabla\frac{1}{|\vec{R}-\vec{R}'|}\,dS = \int_S [\vec{g}(\vec{R})-\vec{g}(\vec{r})]\times\nabla\frac{1}{|\vec{R}-\vec{r}|}\,dS \pm 2\pi\vec{g}(\vec{r})\times\hat{n}(\vec{r}) +$$

$$+ \vec{g}(\vec{r})\times\int_S \hat{n}(\vec{R})\,\frac{\partial}{\partial n}\,\frac{1}{|\vec{R}-\vec{r}|}\,dS + \vec{g}(\vec{r})\times\int_S \nabla_0\,\frac{1}{|\vec{R}-\vec{r}|}\,dS\,. \tag{A.16}$$

Using the argument we employed for Theorems A.3 and A.4, we have

<u>Theorem A.5</u>: If the vector-function $\vec{g}$ has components which are H-regular density functions for $\overline{S}$, then the function

$$\vec{w}(\vec{R}') = \int_S \vec{g}(\vec{R}) \times \nabla G(\vec{R}|\vec{R}')\,dS$$

is continuous at every point $\vec{r} \in S$ in the approach to it from either the side S_+ or the side S_-. Moreover,

$$\int_{S_+(S_-)} \vec{g}(\vec{R}) \times \nabla G(\vec{R}|\vec{R}')\,dS = \pm\frac{1}{2}\,\hat{n}(\vec{r}) \times \vec{g}(\vec{r}) + \tag{A.17}$$

$$+ \int_S \vec{g}(\vec{r}) \times \nabla_0 \frac{1}{|\vec{R}-\vec{r}|}\,dS + \int_S [\vec{g}(\vec{R}) - \vec{g}(\vec{r})] \times \nabla \frac{1}{|\vec{R}-\vec{r}|}\,dS,$$

where the plus sign corresponds to the approach from S_+, and the minus from S_-.

Appendix B. An alternate derivation of the radiation conditions.

In this appendix we prove in a different way than in Section F that (68) and (69) satisfy the radiation conditions. For $R' \to \infty$, we write

$$G(\vec{R}|\vec{R}') = g(\vec{R}|\vec{R}')+0\left(\frac{1}{R'^2}\right), \quad \nabla' G(\vec{R}|\vec{R}') = ikg(\vec{R}|\vec{R}')\hat{R}'+0\left(\frac{1}{R'^2}\right), \tag{B.1}$$

where

$$g(\vec{R}|\vec{R}') = -\frac{e^{ik(R'-\hat{R}'\cdot\vec{R})}}{4\pi R'}, \tag{B.2}$$

and where (B.1) are uniformly valid for all $\vec{R} \in S$ and all $\hat{R}'$. For (69) we then have that

$$\vec{H}(\vec{R}') = \frac{e^{ikR'}}{R'}\vec{A} + 0\left(\frac{1}{R'^2}\right), \quad R' \to \infty \tag{B.3}$$

with

$$\vec{A} = -\frac{ik}{4\pi}\int_S \hat{R}'\times\vec{K}(\vec{R})e^{-ik\hat{R}'\cdot\vec{R}}dS. \tag{B.4}$$

For (68) we have, using (74),

$$\vec{E}(\vec{R}') = -\frac{Z}{ik}\int_S \vec{K}(\vec{R})\cdot\nabla'\nabla' G(\vec{R}|\vec{R}')dS+ikZ\int_S G(\vec{R}|\vec{R}')\vec{K}(\vec{R})dS. \tag{B.5}$$

But, for $R' \to \infty$,

$$\nabla'\nabla' G(\vec{R}|\vec{R}') = ik\nabla' g(\vec{R}|\vec{R}')\hat{R}'+0\left(\frac{1}{R'^2}\right) = -k^2 g(\vec{R}|\vec{R}')\hat{R}'\hat{R}'+0\left(\frac{1}{R'^2}\right), \tag{B.6}$$

so that

$$\vec{E}(\vec{R}') = ikZ \int_S \hat{R}'(\hat{R}'\cdot\vec{K})gdS + ikZ \int_S g\vec{K}dS + 0\left(\frac{1}{R'^2}\right) =$$

$$= ikZ\hat{R}'\times\int_S [\vec{K}(\vec{R})\times\hat{R}']gdS + 0\left(\frac{1}{R'^2}\right) =$$

$$= -Z\,\frac{e^{ikR'}}{R'}\,\hat{R}'\times\vec{A} + 0\left(\frac{1}{R'^2}\right)\ ,\quad R' \to \infty. \tag{B.7}$$

From (B.3) and (B.7) the radiation conditions (10) follow immediately.

Appendix C. The estimation of an integral.

In this appendix we will estimate the integral

$$\int_{\Sigma_\varepsilon} \hat{n}\cdot(\vec{E}^*\times\vec{H})\,dS, \tag{C.1}$$

with Σ_ε defined in (40), which appears in Section G. To this end we choose a point M_0 of C and with it as origin we erect a rectangular coordinate system with the x-axis in the direction of the tangent vector to C at M_0, and the z-axis in the direction of the normal to $\bar{S}$ at M. The y-axis then lies on the plane tangent to $\bar{S}$ at M and is directed toward $\bar{S}$. On the yz-plane we erect a polar coordinate system (ρ',ϕ') as described in Section B. The angle ϕ' is measured counter-clockwise from the positive y-axis. On this plane we draw the circle $\rho' = \varepsilon > 0$ and we estimate the integrand of (C.1) at points of this circle except at the point which belongs to the surface.

Since the unit normal $\hat{n}$ is pointed toward M, we have that

$$\hat{n}\cdot(\vec{E}^*\times\vec{H}) = -\hat{\rho}'\cdot(\vec{E}^*\times\vec{H}) = -(\hat{y}\cos\phi'+\hat{z}\sin\phi')\cdot(\vec{E}^*\times\vec{H}) =$$

$$= -\cos\phi'(\vec{E}^*\times\vec{H})_y - \sin\phi'(\vec{E}^*\times\vec{H})_z =$$

$$= -\cos\phi'(E_z{}^*H_x - E_x{}^*H_z) - \sin\phi'(E_x{}^*H_y - E_y{}^*H_x) =$$

$$= E_x{}^*(\cos\phi' H_z - \sin\phi' H_y) + H_x(\sin\phi' E_y{}^* - \cos\phi' E_z{}^*) \tag{C.2}$$

As mentioned above, the integral in (C.1) appears in Section G where the functions $\vec{E}$ and $\vec{H}$ are assumed to be solutions of the open surface problem. Then, by the equivalence theorem 7, $\vec{E}$ and $\vec{H}$ are given by (68) and (69), respectively. These expressions appear rewritten in (85) and (86) in a way so as to isolate the singularity of G. As pointed out in the disucssion there, in examining the behavior of $\vec{E}$ and $\vec{H}$ near the edge we need only cosider the first integrals in (85) and (86). Thus, with $\vec{R}' = (\rho',\phi')$, we have

$$\vec{E}(\vec{R}') = -\frac{1}{4\pi}\int_S \left[\frac{Z}{ik}\nabla_0\cdot\vec{K}(\vec{R})\nabla\frac{1}{|\vec{R}-\vec{R}'|} + \frac{ikZ}{|\vec{R}-\vec{R}'|}\vec{K}(\vec{R})\right]dS + 0(1) \qquad (C.3)$$

$$\vec{H}(\vec{R}') = -\frac{1}{4\pi}\int_S \vec{K}(\vec{R})\times\nabla\frac{1}{|\vec{R}-\vec{R}'|}\,dS + 0(1). \qquad (C.4)$$

We will estimate (C.2) by using these last two expressions. We note that the densities appearing in them are H-regular density functions for $\overline{S}$ by condition (i) in Theorem 6 so that we can use the results in Part I. Though $\vec{K}$ and $\nabla_0\cdot\vec{K}$ are complex-valued we will treat them as real-valued, the extension to complex-values being obvious. From Theorem 1, Part I, we have that

$$E_x(\vec{R}') = 0(1), \qquad \rho' \to 0. \qquad (C.5)$$

We will now prove that the same is true for H_x. From (C.4) and with respect to the coordinate system above, we write

$$H_x(\vec{R}') = -\frac{1}{4\pi}\int_S \left[K_y(\vec{R})\frac{\partial}{\partial z}\frac{1}{|\vec{R}-\vec{R}'|} - K_z(\vec{R})\frac{\partial}{\partial y}\frac{1}{|\vec{R}-\vec{R}'|}\right] dS + 0(1)$$

$$= \frac{1}{4\pi}\int_S [K_y(\vec{R}')(z-z')-K_z(\vec{R})(y-y')]\frac{dS}{|\vec{R}-\vec{R}'|^3} + 0(1). \qquad \text{(C.6)}$$

We note that the integral here is a combination of the integrals appearing in (I.43) and (I.74) and will be treated in the same way up to a certain point. As remarked at the beginning of Section D, Part I, the integration over S can be split into two parts one over $S(\Lambda)$ and one over $S-S(\Lambda)$, where $S(\Lambda)$ is that neighborhood of S about M_0 whose projection on the xy-plane is the region Λ defined in (I.12). The integral over $S-S(\Lambda)$ is continuous in a neighborhood of M_0, and its limit as $\rho' \to 0$ is equal to the analogous integral obtained when $\vec{R}$ is replaced by $\vec{0}$. The integral over $S(\Lambda)$ can be written as

$$\int_{S(\Lambda)} [K_z(\vec{R})(y-y')-K_y(\vec{R})(z-z')]\frac{dS}{|\vec{R}-\vec{R}'|^3} =$$

$$= \int_\Lambda \frac{K_z(x,y)(y-y')\sec\psi(x,y)}{|\vec{R}-\vec{R}'|^3}dxdy -$$

$$\int_\Lambda \frac{K_z(x,y)(y-y')\sec\psi(x,y)}{|\vec{R}-\vec{R}'|^3}dxdy = I_1-I_2, \qquad \text{(C.7)}$$

where ψ is defined in (I.25) and is the angle the z-axis makes with the normal to the surface at (x,y).

For I_1 we write

$$I_1 = \int_\Lambda [K_z(x,y)\sec\psi(x,y) - K_z(0,y)\sec\psi(0,y)] \frac{y-y'}{|\vec{R}-\vec{R}'|^3} dxdy +$$

$$+ \int_\Lambda K_z(0,y)\sec\psi(0,y)(y-y')\left[\frac{1}{|\vec{R}-\vec{R}'|^3} - \frac{1}{R^3}\right] dxdy +$$

$$+ \int_\Lambda K_z(0,y)\sec\psi(0,y) \frac{y-y'}{R^3} dxdy, \tag{C.8}$$

where

$$R = \sqrt{x^2+(y-y')^2+(F(0,y)-z')^2} \quad . \tag{C.9}$$

The second integral above has been discussed in Part I, eqs. (I.32)-(I.36), and has been found to be bounded. For the first integral, call it I_3, we write

$$I_3 = \int_\Lambda [K_z(x,y)-K_z(0,y)]\sec\psi(x,y) \frac{y-y'}{|\vec{R}-\vec{R}'|^3} dxdy +$$

$$+ \int_\Lambda [\sec\psi(x,y)-\sec\psi(0,y)]K_z(0,y) \frac{y-y'}{|\vec{R}-\vec{R}'|^3} dxdy \; . \tag{C.10}$$

The second integral here can be shown to be bounded by treating it in the same way as the one appearing in (I.24). For the first integral we use the mean value theorem to write

$$K_z(x,y)-K_z(0,y) = \left(\frac{\partial K_z(x,y)}{x}\right)_{x=x_1} x \quad , \quad 0 \lesssim x \lesssim x_1 . \tag{C.11}$$

Since $\partial K_z/\partial x$ is an H-regular density function for $\bar{S}$, the integral reduces to one as in (I.29) which has been shown to be bounded.

For I_2 we write

$$I_2 = \int_\Lambda [K_y(x,y)\sec\psi(x,y) - K_y(0,y)\sec\psi(0,y)] \frac{z-z'}{|\vec{R}-\vec{R}'|^3}\, dxdy\ +$$

$$+ \int_\Lambda K_y(0,y)\sec\psi(0,y)(z-z')\left[\frac{1}{|\vec{R}-\vec{R}'|^3} - \frac{1}{R^3}\right] dxdy\ +$$

$$+ \int_\Lambda K_y(0,y)\sec\psi(0,y)\ \frac{F(x,y)-F(0,y)}{R^3}\, dxdy\ +$$

$$+ \int_\Lambda K_y(0,y)\sec\psi(0,y)\ \frac{F(0,y)-z'}{R^3}\, dxdy. \qquad (C.12)$$

The first two integrals in this expression can be treated as the corresponding ones for I_1. The third integral is of the same type as the one appearing in (I.76) and is bounded. We can then write

$$I_1 - I_2 = \int_\Lambda [K_z(0,y)(y-y') - K_y(0,y)(F(0,y)-z')] \frac{\sec\psi(0,y)}{R^3} dxdy + O(1),$$
$$\rho' \to 0. \qquad (C.13)$$

In order to estimate this integral we will first rewrite it slightly differently. We note that K_z and K_y are evaluated at points of S which also belong to the yz-plane, i.e. on the curve $z = F(0,y)$. As explained in Part I, after eq. (I.2), the normal to the surface at such points and the normal to the curve coincide because of condition (I.2). This normal is given by

$$\hat{n} = \frac{-F_y(0,y)\hat{y}+\hat{z}}{\sqrt{1+F_y^{\,2}(0,y)}} . \tag{C.14}$$

The unit tangent vector to the curve is

$$\hat{T} = \frac{\hat{y}+F_y(0,y)\hat{z}}{\sqrt{1+F_y^{\,2}(0,y)}} , \tag{C.15}$$

so that $\hat{T}\times\hat{n} = \hat{x}$. Since the current density $\vec{K}$ is a surface field, it can be written as

$$\vec{K}(0,y) = K_T(0,y)\hat{T}+K_x(0,y)\hat{x}, \tag{C.16}$$

so that

$$K_y(0,y) = \frac{K_T(0,y)}{\sqrt{1+F_y^{\,2}(0,y)}} , \qquad K_z(0,y) = \frac{F_y(0,y)K_T(0,y)}{\sqrt{1+F_y^{\,2}(0,y)}} , \tag{C.17}$$

and, since $\sec\psi(0,y) = \sqrt{1+F_y^{\,2}(0,y)}$, we have for (C.13)

$$I_1-I_2 = \int_\Lambda [F_y(0,y)(y-y')-(F(0,y)-z')]\,\frac{K_T(0,y)}{R^3}\,dxdy+0(1), \quad \rho' \to 0$$

We now employ the coordinates (s,ρ) and use the results for the integrals in (I.45) and (I.78) to get

$$I_1 - I_2 = 2\int_0^b \frac{K_T(0,\rho)(1-\tilde{\kappa}\rho)}{(\rho-y')^2+(F(0,\rho)-z')^2}\,[F_\rho(0,\rho)(\rho-y')-(F(0,\rho)-z')]d\rho + 0(1), \tag{C.19}$$

where $\tilde{\kappa} = \kappa(0)$, κ being the curvature of the projection of the boundary C on the xy-plane defined in (I.11). With

$$u = \rho + i(F(0,\rho), \quad w = y'+iz' = \rho'(\cos\phi'+i\sin\phi') \tag{C.20}$$

we can write

$$I_1 - I_2 = 2\,\mathrm{Im}\int_{C_1} \frac{K_T(0,\rho)(1-\tilde{\kappa}\rho)}{u-w}\,du + 0(1) \tag{C.21}$$

where C_1 is the curve whose equation is $z = F(0,\rho)$, $0 \le \rho \le b$. Since K_T is Hölder-continuous on C_1, including the end-points, and since $K_T(0,0) = 0$, we have that the integral is a bounded function of w tending to a definite limit as w approaches the end-point $(0,0)$ along any path (Muskhelishvili, 1953). From (C.21), (C.7), and (C.6) we have that

$$H_x(\vec{R}') = 0(1), \quad \rho' \to 0, \tag{C.22}$$

so that (C.2) becomes

$$|\hat{n}\cdot(\vec{E}^\star\times\vec{H})| \le A(|H_y|+|H_z|+|E_y|+|E_z|) \le A(|\vec{H}|+|\vec{E}|), \quad \rho' \to 0, \tag{C.23}$$

where A is a positive real number.

From the Cauchy-Schwarz inequality and (C.23), we have for (C.1)

$$\left|\iint_{\Sigma_\varepsilon} \hat{n}\cdot(\vec{E}^*\times\vec{H})\,dS\right|^2 \leq \left(\iint_{\Sigma_\varepsilon} |\hat{n}\cdot(\vec{E}^*\times\vec{H}|\,dS\right)^2 =$$

$$= \left(\int_0^L\int_0^{2\pi} |\hat{n}\cdot\vec{E}^*\times\vec{H}|\ \varepsilon\left(1-\varepsilon\hat{\rho}'\cdot\frac{d\hat{t}}{ds'}\right)d\phi'ds'\right)^2 \leq$$

$$\leq \varepsilon\left\{\int_0^L\int_0^{2\pi}\left(1-\varepsilon\hat{\rho}'\cdot\frac{d\hat{t}}{ds'}\right)^2 d\phi'ds'\right\}\left\{\int_0^L\int_0^{2\pi}|\hat{n}\cdot\vec{E}^*\times\vec{H}|^2 d\phi'ds'\right\} \leq$$

$$\leq A'\varepsilon^2 \int_0^L \int_0^{2\pi} (|\vec{E}|^2+|\vec{H}|^2)\,d\phi'ds' \ , \tag{C.24}$$

where A' is a positive real number. Combining this result with (11), we get

$$\int_{\Sigma_\varepsilon} \hat{n}\cdot(\vec{E}^*\times\vec{H})\,dS = 0(\varepsilon^\alpha), \qquad \varepsilon \to 0. \tag{C.25}$$

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR - TR - 77 - 0626	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ELECTROMAGNETIC SCATTERING BY PERFECTLY CONDUCTING OPEN SURFACES		5. TYPE OF REPORT & PERIOD COVERED Interim 6. PERFORMING ORG. REPORT NUMBER 2634-4
7. AUTHOR(s) John S. Asvestas		8. CONTRACT OR GRANT NUMBER(s) AFOSR 74-2634
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics University of Delaware Newark, Delaware 19711		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research /NM Bolling AFB Washington, D.C. 20332		12. REPORT DATE March 1977 13. NUMBER OF PAGES 104
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Open surface scattering
Edge currents
Electromagnetic integral equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The behavior of simple layer potentials and their spatial derivatives near the edge of an open surface is analyzed. Conditions are determined on the surface geometry and on the density distributions for which the potentials have locally finite energy. These results are applied to the formulation problems of electromagnetic scattering from open surfaces as integral equations. →

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20. Abstract

It is shown that for certain classes of open surfaces and current densities, the boundary value problem is equivalent to a problem in integral equations of the first kind which can have at most one solution.